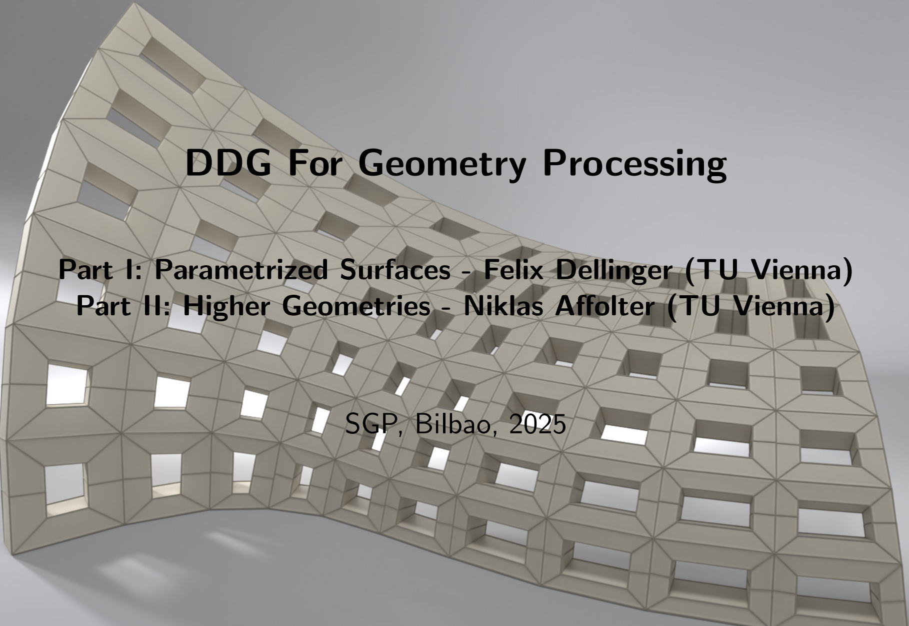
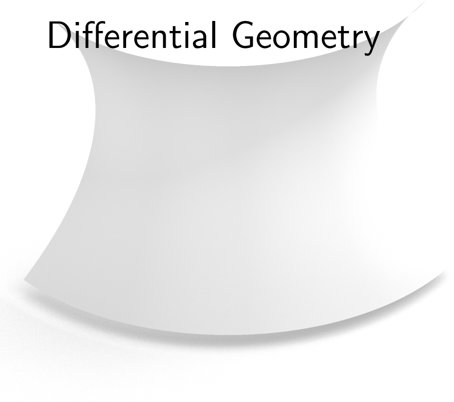


DDG For Geometry Processing

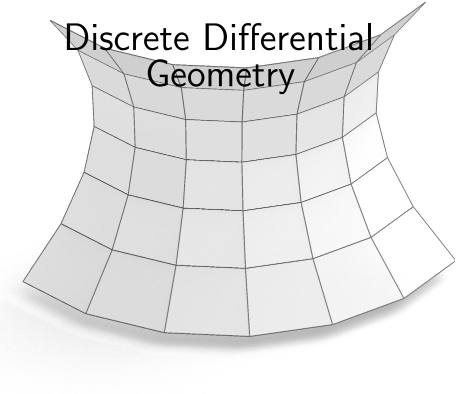
Part I: Parametrized Surfaces - Felix Dellinger (TU Vienna)
Part II: Higher Geometries - Niklas Affolter (TU Vienna)

SGP, Bilbao, 2025

Differential Geometry



Discrete Differential Geometry



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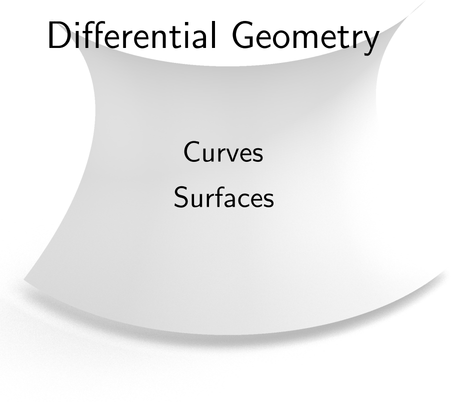
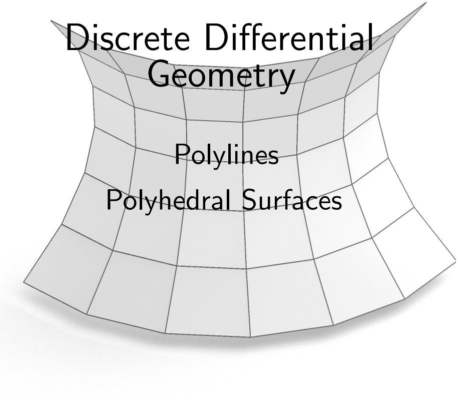


Curves

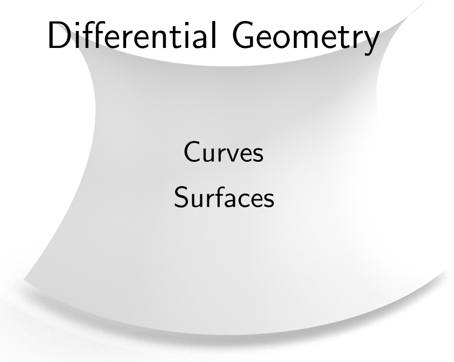
Surfaces

Discrete Differential Geometry



Polylines

Polyhedral Surfaces



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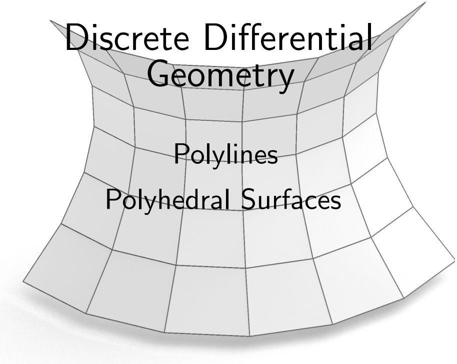
Surfaces



Discrete Differential Geometry

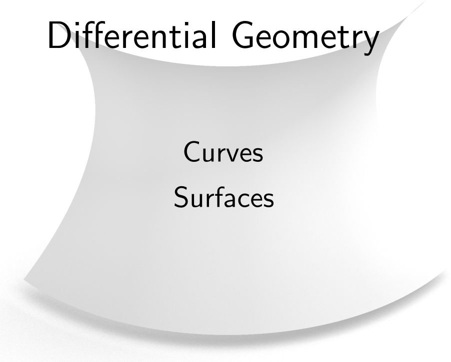
Polylines

Polyhedral Surfaces

Goal

Develop a geometric theory based on discrete objects.

Differential Geometry



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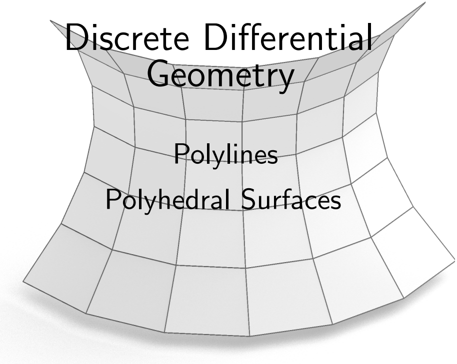
Surfaces

Discrete Differential Geometry



Polylines

Polyhedral Surfaces

Goal

Develop a geometric theory based on discrete objects.

How can we use it?

- Discrete formulations can easily be turned into code
- Obtain visually or structurally optimized meshes
- Form-finding through local mesh constraints

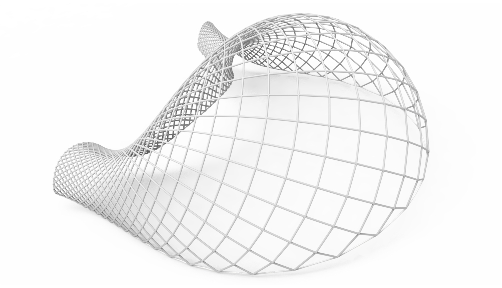
Roadmap

Objectives

- Planar and orthogonal faces
- Invariant mesh properties
- Offset structures
- Developable and minimal surfaces

Methods

- Optimization
- Smooth curves
- Transformations
- Subdivision



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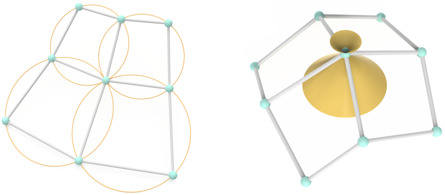

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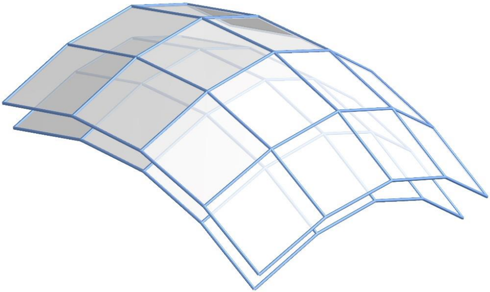

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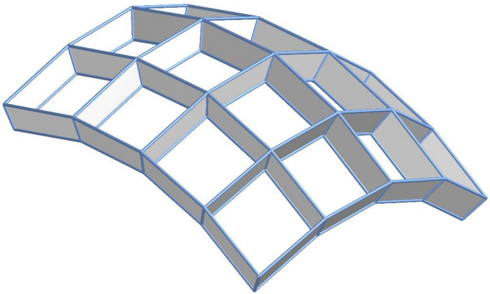

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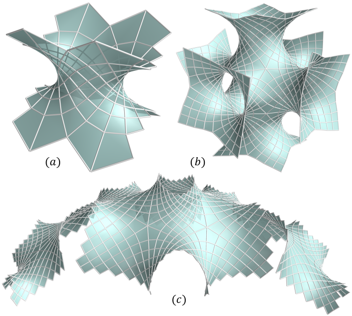
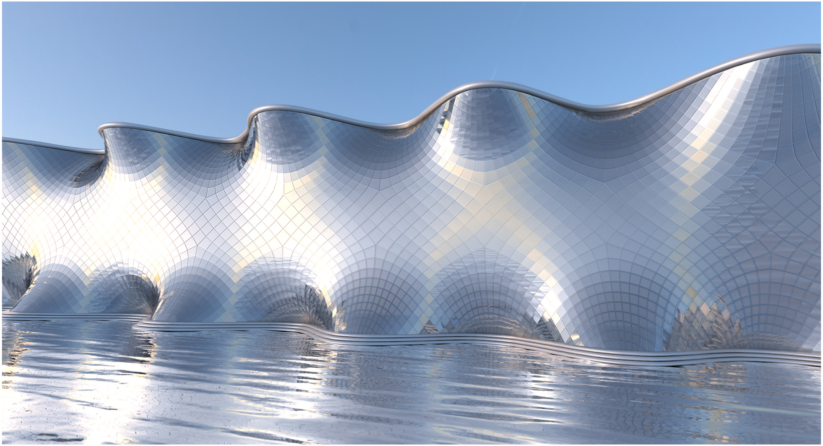
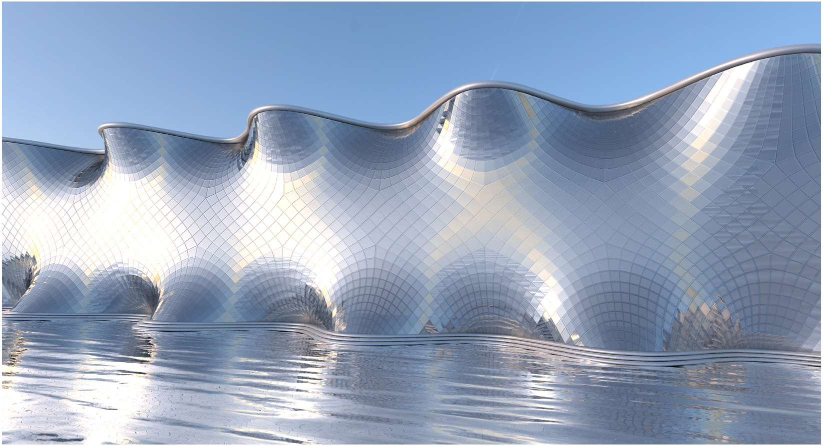


Planar Quadrilaterals and Conjugate Curves



Planar Quadrilaterals and Conjugate Curves



How do we find the right quad mesh on a given shape?

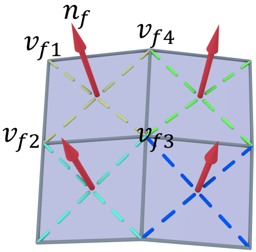
Planar Quadrilaterals and Conjugate Curves

Total Energy

$$E = E_{PQ} + \omega E_{fair}$$

Energy term for planarity

$$E_{PQ} = \sum_{f=1}^{|F|} \sum_{j=1}^{4} (n_f \cdot (v_{fj} - v_{fj-1}))^2 + \sum_{f=1}^{|F|} (n_f \cdot n_f - 1)^2$$



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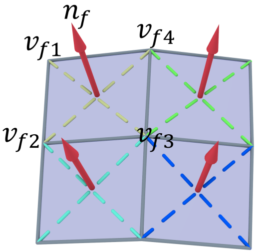
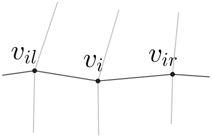


Fairness Energy term

$$E_{Fair} = \sum_{i \in polyline} (2v_i - v_{il} - v_{ir})^2$$



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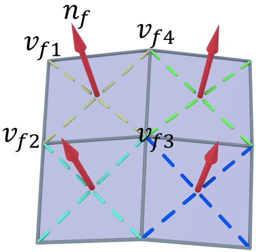

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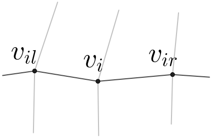
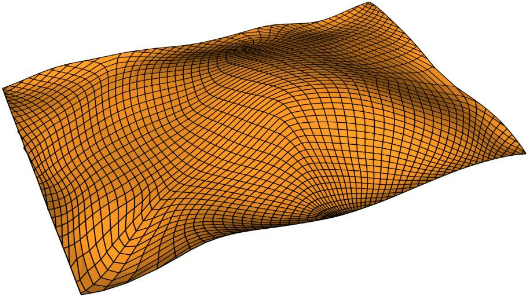
$$E_{Fair} = \sum_{i \in polyline} (2v_i - v_{il} - v_{ir})^2$$

$\omega \approx 0.001$



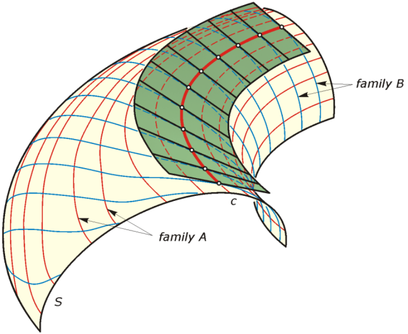
Planar Quadrilaterals and Conjugate Curves



Theorem

A quadrilateral mesh with planar faces is a discrete version of a conjugate net of curves.

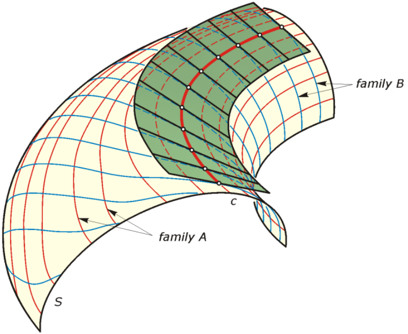
Planar Quadrilaterals and Conjugate Curves



Definition of a conjugate net

The tangents of the curves of family A along any curve of family B form a developable surface. (And vice versa)

Planar Quadrilaterals and Conjugate Curves



Definition of a conjugate net

The tangents of the curves of family A along any curve of family B form a developable surface. (And vice versa) I.e. Tangents intersect their infinitesimal neighbor.

Computing Conjugate Curves

Local Parametrization

$$f:\ \mathbb{R}^2 \to \mathbb{R}^3$$

$f_u, f_v \ldots$ partial derivatives

$n := f_u \times f_v \ldots$ normal vector

Computing Conjugate Curves

Local Parametrization

$$f:\ \mathbb{R}^2 \to \mathbb{R}^3$$

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$n := f_u \times f_v \ldots$ normal vector

Fundamental Forms

$$\mathrm{I} = \begin{pmatrix} \langle f_u, f_u \rangle & \langle f_u, f_v \rangle \\ \langle f_v, f_u \rangle & \langle f_v, f_v \rangle \end{pmatrix} \qquad \mathrm{II} = \begin{pmatrix} \langle f_{uu}, n \rangle & \langle f_{uv}, n \rangle \\ \langle f_{vu}, n \rangle & \langle f_{vv}, n \rangle \end{pmatrix}$$

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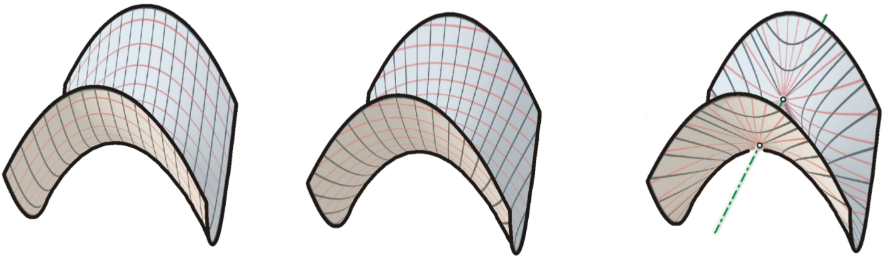
Conjugate Directions

Directions a and b in the parameter domain are conjugate $\Leftrightarrow a^T \,\mathrm{II}\, b = 0$.

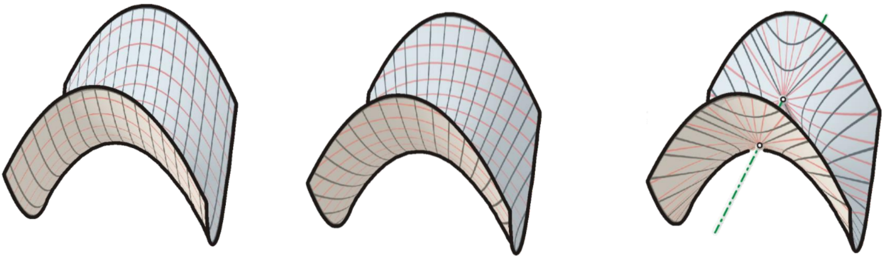
Conjugate Curves Example



Idea

You can choose one family of curves and then compute the second family by integrating the vector field of the conjugate directions.

Conjugate Curves Example



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Warning

The quads can become arbitrarily acute if the curves get close to asymptotic (i.e. self-conjugate) curves.

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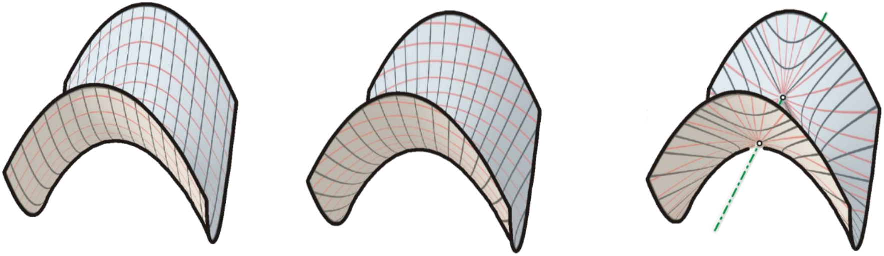


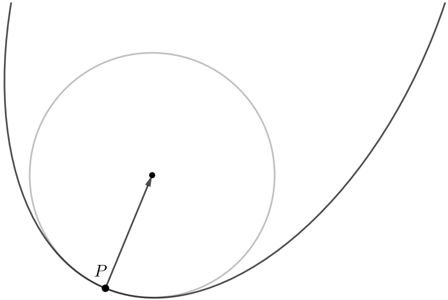
Idea

You can choose one family of curves and then compute the second family by integrating the vector field of the conjugate directions.

Warning

The quads can become arbitrarily acute if the curves get close to asymptotic (i.e. self-conjugate) curves. $\Rightarrow$ Use principal directions

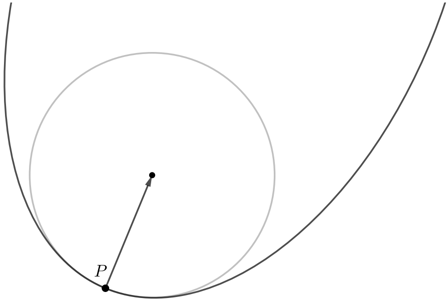
Orthogonal Faces and Principal Curvature Lines



Curvature of a planar curve

The curvature is one over the radius of the osculating circle. This is the circle that approximates the curve best.

Orthogonal Faces and Principal Curvature Lines



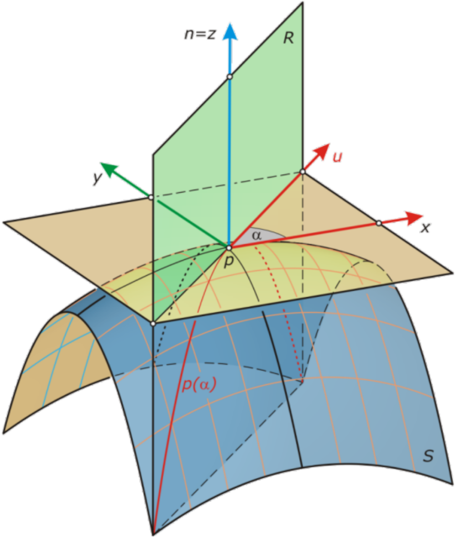
Curvature of a planar curve

The curvature is one over the radius of the osculating circle. This is the circle that approximates the curve best. The plane containing the osculating circle is the osculating plane.

Orthogonal Faces and Principal Curvature Lines

Definition

Principal curvature (pc) lines are the curves that follow the directions of maximal/minimal normal curvature in a surface.



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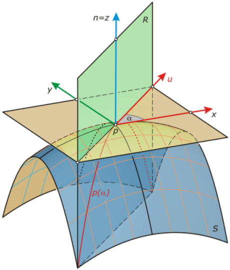

Properties

- PC lines are conjuagte and orthogonal
- Any net of conjugate and orthogonal curves is the net of pc lines.
- The pc directions are the eigenvectors of the shape operator.
- The normals along pc lines form developable surfaces.

Computing Principal Curvature Lines

Recall the fundamental forms

$$I = \begin{pmatrix} \langle f_u, f_u \rangle & \langle f_u, f_v \rangle \\ \langle f_v, f_u \rangle & \langle f_v, f_v \rangle \end{pmatrix} \qquad II = \begin{pmatrix} \langle f_{uu}, n \rangle & \langle f_{uv}, n \rangle \\ \langle f_{vu}, n \rangle & \langle f_{vv}, n \rangle \end{pmatrix}$$

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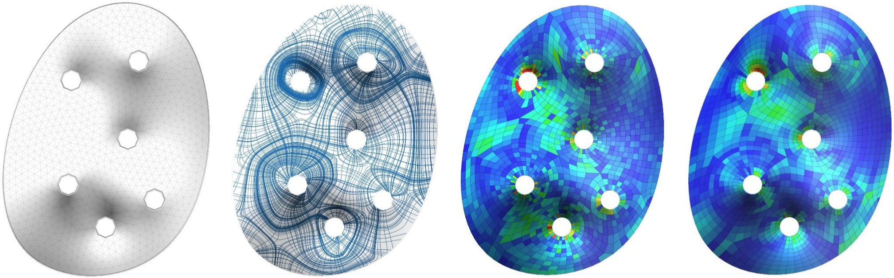
The principal directions in the parameter domain of f are the eigenvectors of

$$S = (\mathrm{I})^{-1}\,\mathrm{II}\,.$$

Principal Remeshing Pipeline



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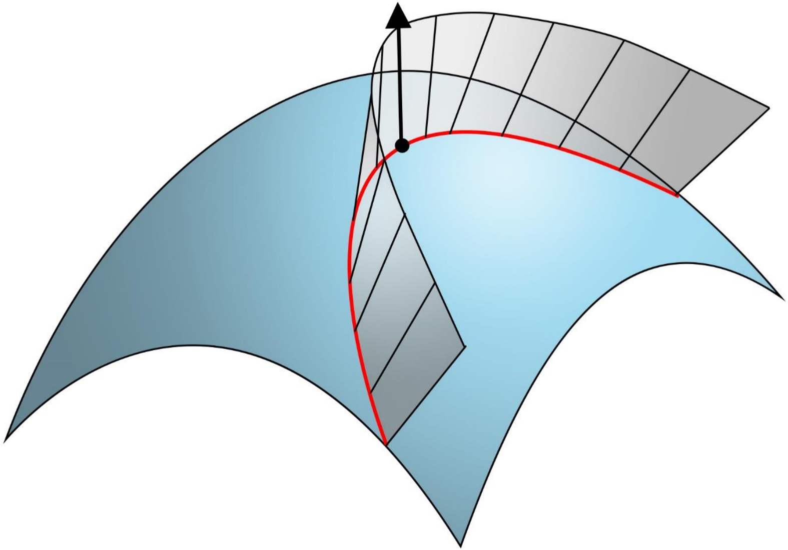
Principal Curvature Lines



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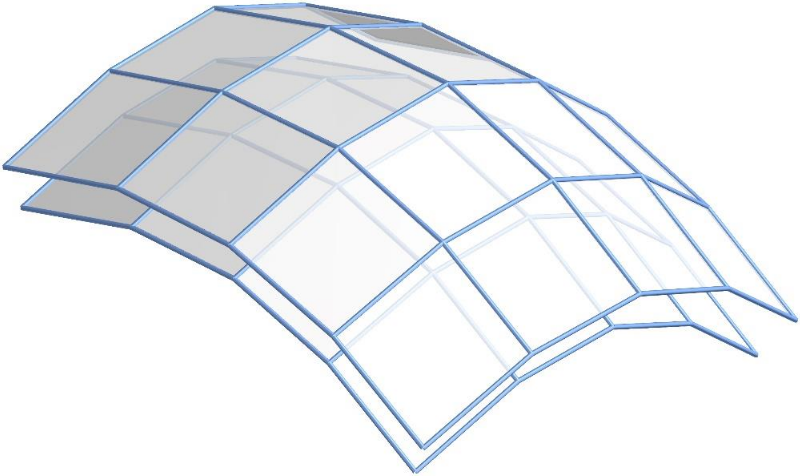

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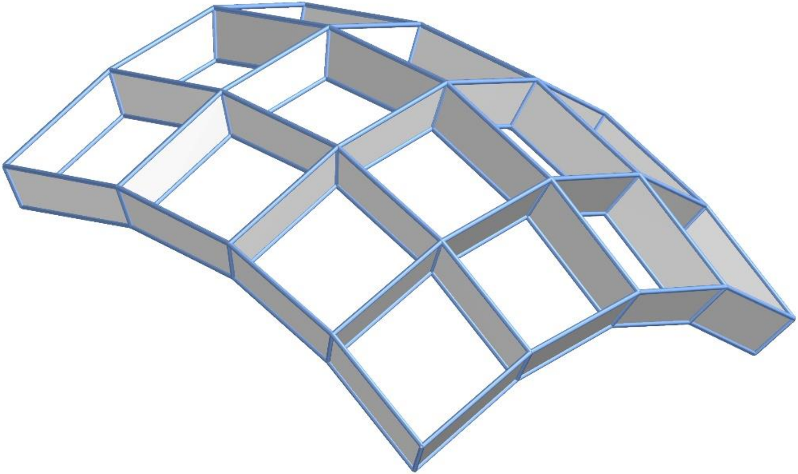
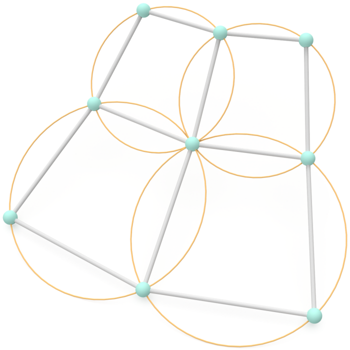
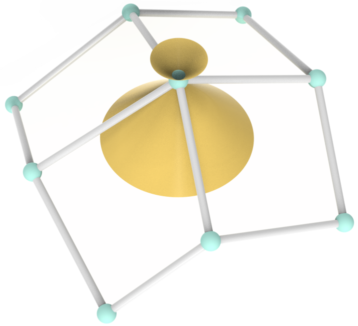


Principal Curvature Meshes

Circular Meshes



Conical Meshes



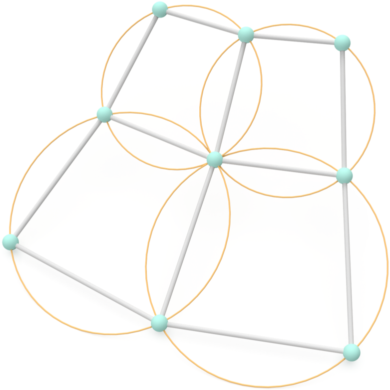
Circular Meshes

Definition

A quad mesh where every face has a circumcircle.

Properties

- Invariant under Moebius transformations
- The sum of opposite angles in a face equals π.
- Allow a parallel offset structure at constant vertex distance.



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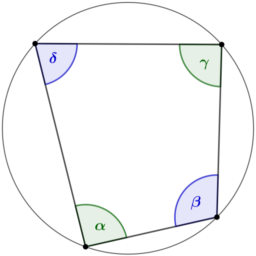

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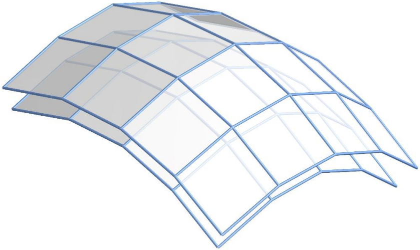

Computing Circular Meshes

Total Energy

$E = E_{circ} + E_{PQ} + \omega E_{fair}$

Energy term for circularity

Use the angle property in every face.

$$E_{circ} = \sum_{f=1}^{|F|} (\omega_{f1} - \omega_{f2} + \omega_{f3} - \omega_{f4})^2$$

Conical Meshes

Definition

All faces that share a vertex touch a common cone.

Properties

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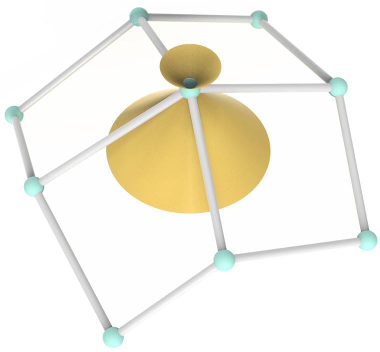

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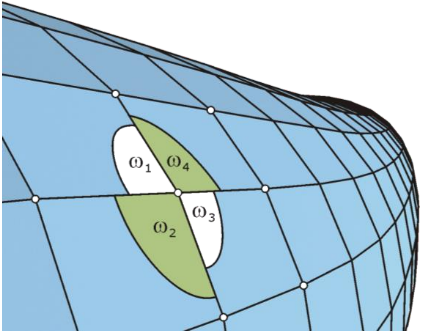

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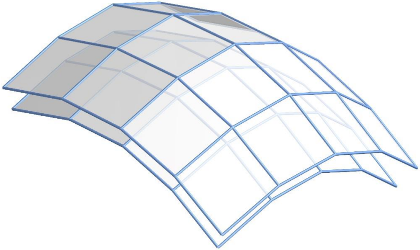


Computing Conical Meshes

Total Energy

$E = E_{cone} + E_{PQ} + \omega E_{fair}$

Energy term for conical meshes

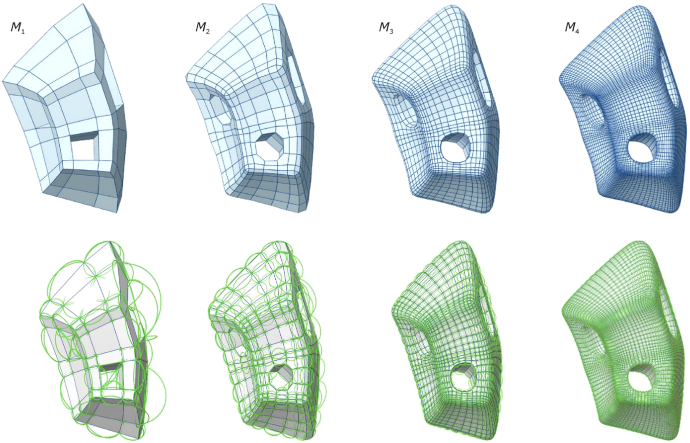
Sum over all inner vertices

$$E_{cone} = \sum_{i=1}^{|V|} (\omega_{i1} - \omega_{i2} + \omega_{i3} - \omega_{i4})^2$$

Design Pipeline: Subdivision

Idea

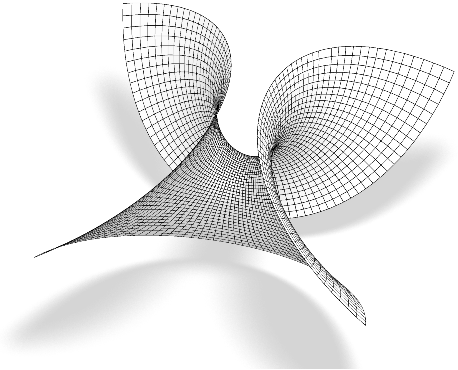
Start with a coarse mesh and alternate between subdivision and feature optimization.



Design Pipeline: Transformations

Idea

Start with a well understood geometry. Compute everything in an invariant way and then transform it.



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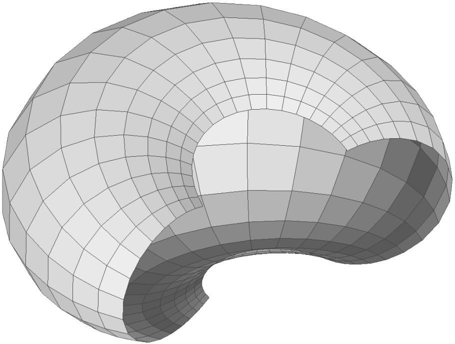

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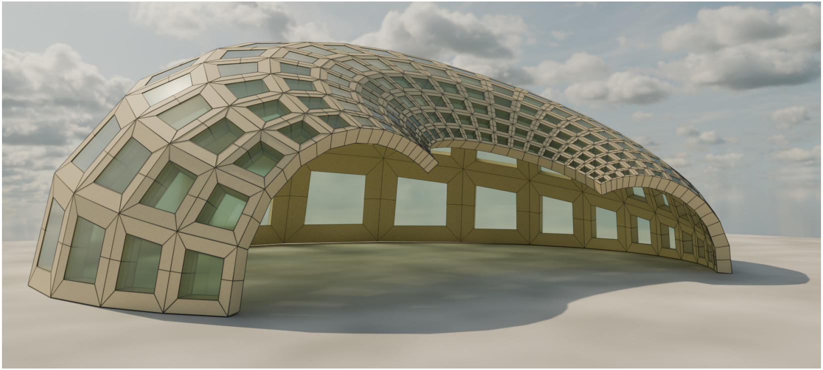
Idea

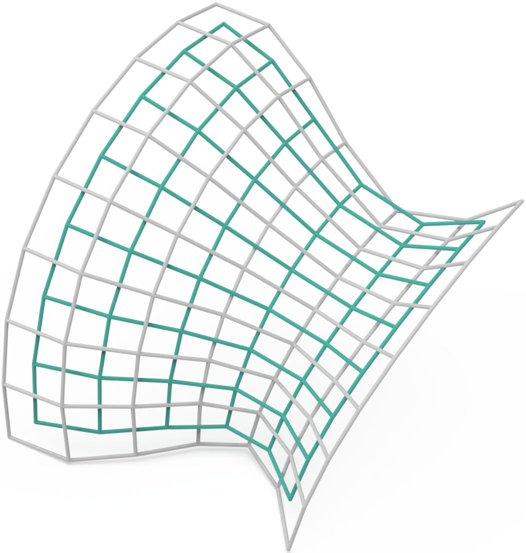
Start with a well understood geometry. Compute everything in an invariant way and then transform it.



What is next?

- Orthogonal Curves
- Geodesics
- Asymptotic Curves

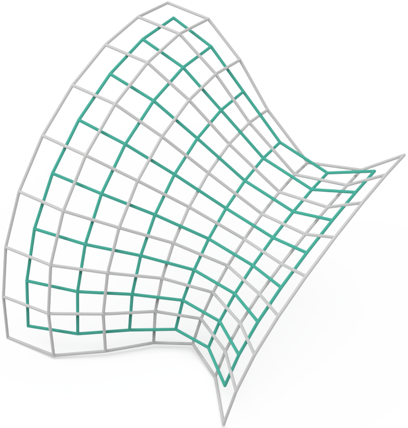
Discrete Orthogonality via Bi-Nets



[Bobenko, Schief, Suris, Techter 2018]

Discrete Orthogonality via Bi-Nets

- Idea: Use two nets to describe the same surface.
- First order properties are encoded in the relation of dual edges.
- Generalizes a lot of existing discretizations.
- Slightly too many meshes...



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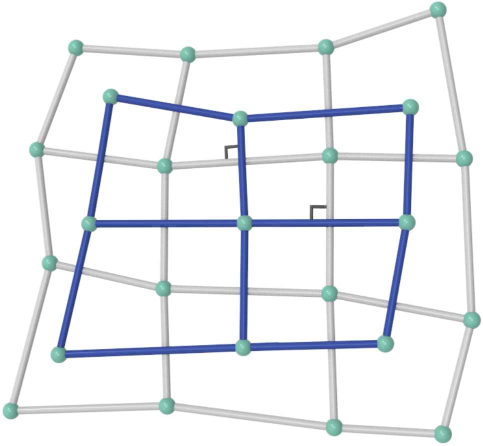

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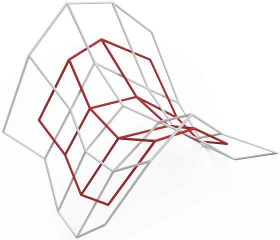
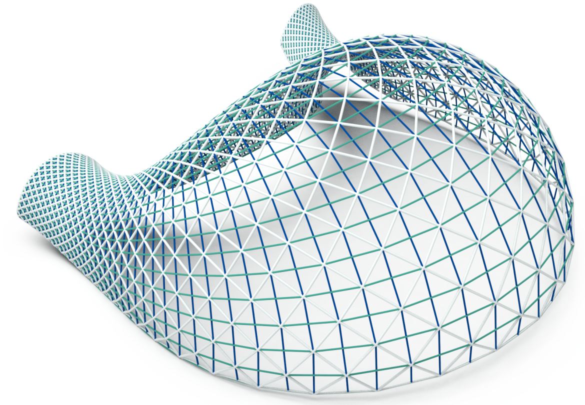


Bi-Nets naturally arise as diagonal nets



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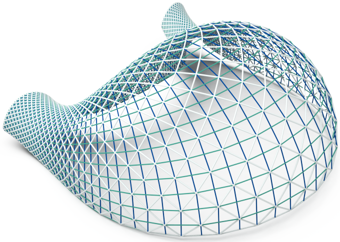
$\phi(u, v) \ldots$ parametrization



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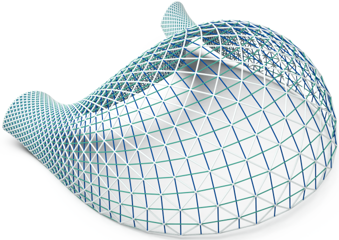
$\psi(u, v) = \phi(u + v, u - v) \ldots$ diag. para.



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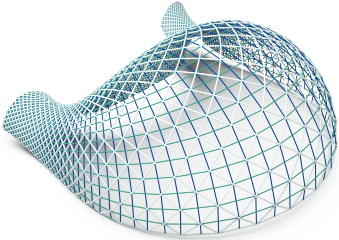
$\partial_1 \psi = \partial_1 \phi + \partial_2 \phi \quad \partial_2 \psi = \partial_1 \phi - \partial_2 \phi$



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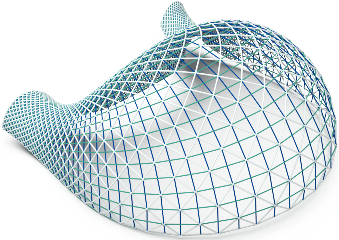
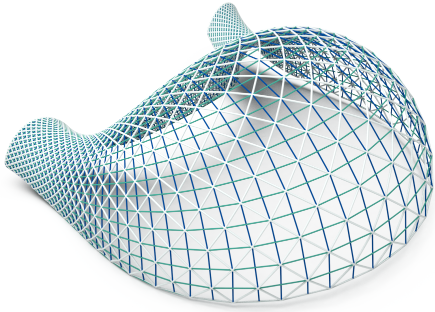
$\|\partial_1\psi\| = \|\partial_2\psi\| \quad \Leftrightarrow \quad \partial_1\phi \perp \partial_2\phi$



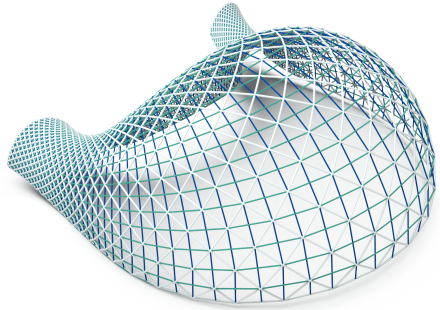
Discrete Orthogonality



Definition

A quadrilateral net is orthogonal if its diagonal nets form a rhombic bi-net.

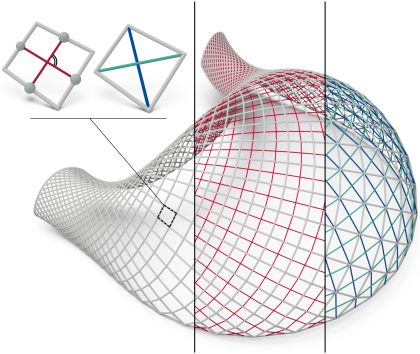
Discrete Orthogonality



Definition

A quadrilateral net is orthogonal if the two diagonals in every quad have equal length. [Wang, Pottmann 2022]

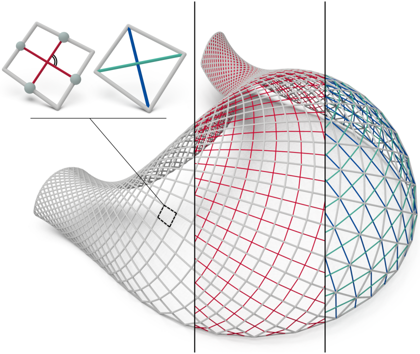
Discrete Orthogonality



Discrete orthogonality

- Defined via equal diagonal lengths

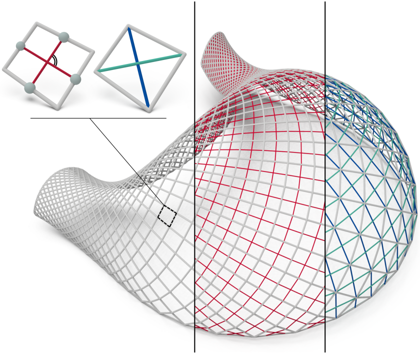
Discrete Orthogonality



Discrete orthogonality

- Defined via equal diagonal lengths
- Observable in the medial lines

Discrete Orthogonality



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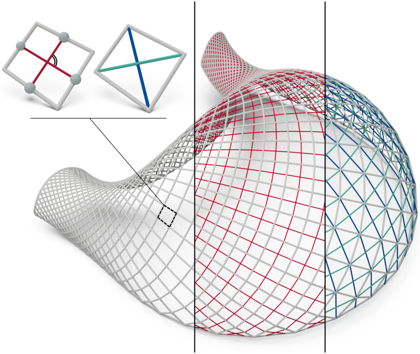
- Observable in the medial lines
- Second order approximation

Discrete Orthogonality



Discrete orthogonality

- Defined via equal diagonal lengths
- Observable in the medial lines
- Second order approximation
- Possible for general quadrilaterals

Applications: Developable Surfaces

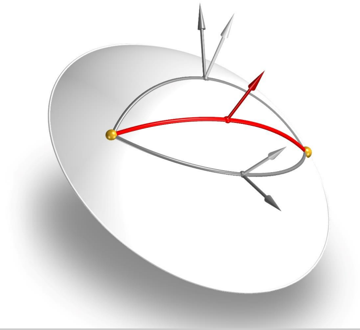
Figure: Walt Disney Concert Hall by Frank O. Gehry

Applications: Developable Surfaces

Applications: Developable Surfaces

Idea

Use orthogonal geodesics. [Rabinovich, Hoffmann, Sorkine-Hornung 2018]



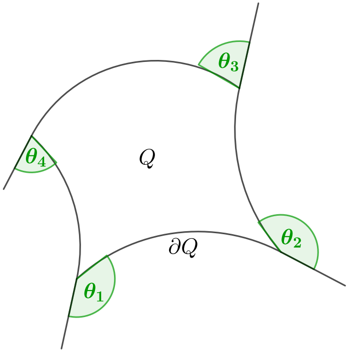
Definition

Geodesics are locally the shortest path between two points. Their osculating plane is orthogonal to the surface.

Applications: Developable Surfaces

Idea

Use orthogonal geodesics. [Rabinovich, Hoffmann, Sorkine-Hornung 2018]



Gauss-Bonnet Theorem

$$\int_Q K \, \mathrm{dA} + \int_{\partial Q} \kappa_g \, \mathrm{ds} = 2\pi - \sum_{i=1}^{4} \theta_i$$

Applications: Developable Surfaces

Total Energy

$E = E_{Ortho.} + \omega_1 E_{fair} + E_{Gnet}$

Applications: Developable Surfaces

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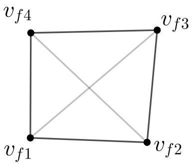

Energy term for orthogonality

$$E_{Ortho.} = \sum_{f=1}^{|F|} \left(\|v_{f1} - v_{f3}\|^2 - \|v_{f2} - v_{f4}\|^2 \right)^2$$

Applications: Developable Surfaces

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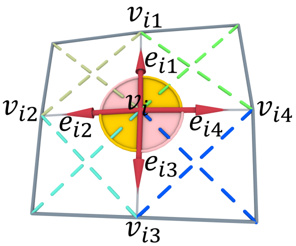
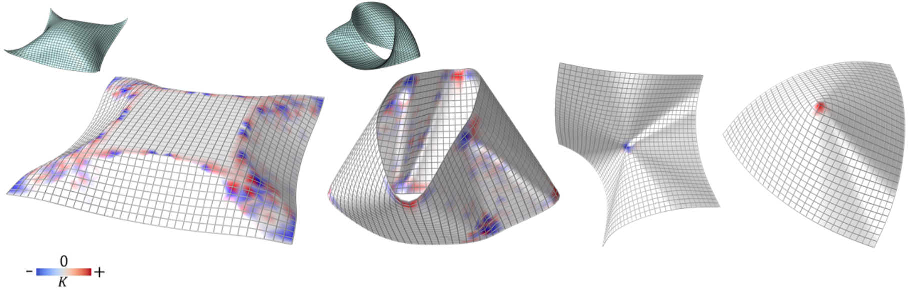


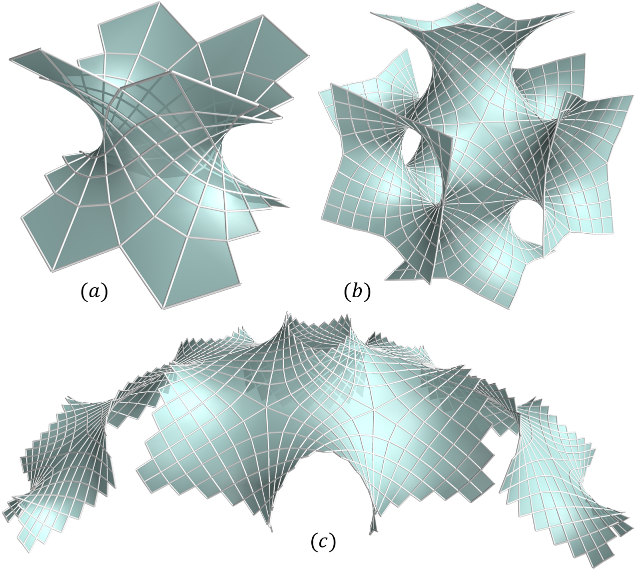
Energy term for geodesics

$$E_{Gnet} = \sum_{i=1}^{|V|}((e_{i1} \cdot e_{i2} - e_{i3} \cdot e_{i4})^2 + (e_{i2} \cdot e_{i3} - e_{i4} \cdot e_{i1})^2) + \sum_{i=1}^{|V|}\sum_{j=1}^{4}\left(e_{ij} - \frac{v_{ij} - v_i}{\|v_{ij} - v_i\|}\right)^2$$

Applications: Developable Surfaces



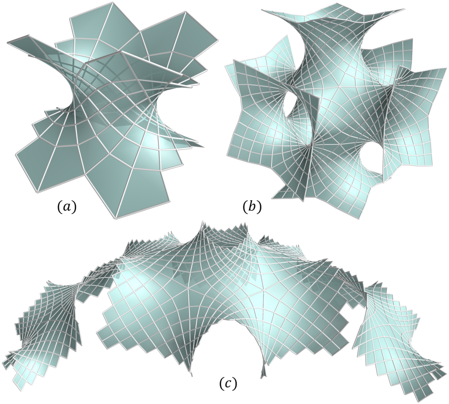
Applications: Minimal Surfaces



Applications: Minimal Surfaces

Idea

Use an orthogonal asymptotic net.



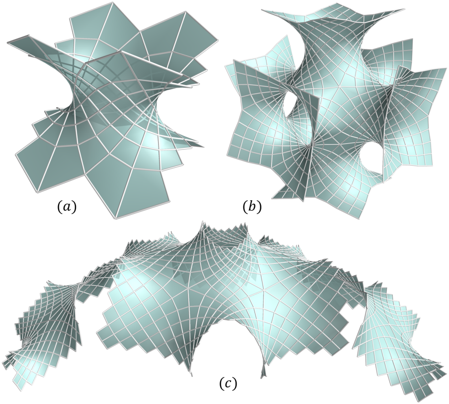
Applications: Minimal Surfaces

Idea

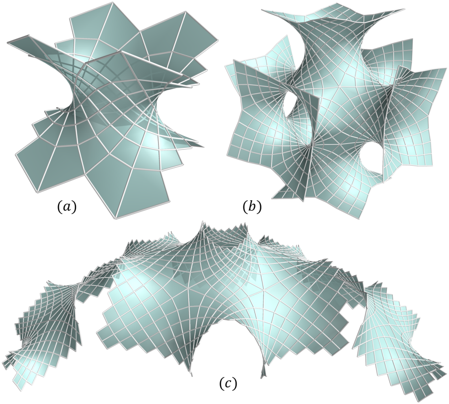
Use an orthogonal asymptotic net.

Definition

A curve is asymptotic $\Leftrightarrow$ its osculating plane is the tangent plane.



Applications: Minimal Surfaces



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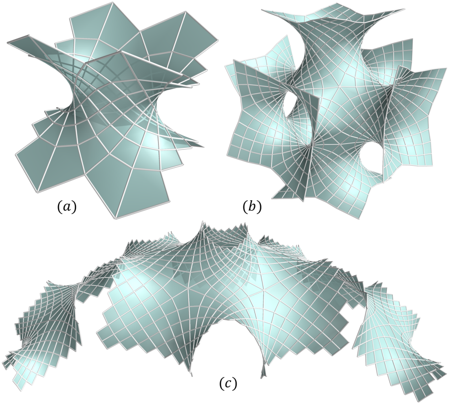
A curve is asymptotic $\Leftrightarrow$ its osculating plane is the tangent plane.

Why is it minimal?

$\tan(\alpha/2)^2 = -\kappa_1/\kappa_2$

Applications: Minimal Surfaces



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A curve is asymptotic $\Leftrightarrow$ its osculating plane is the tangent plane.

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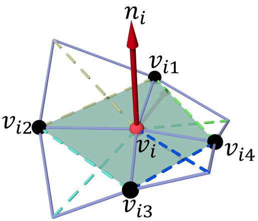
$$\tan(\alpha/2)^2 = -\kappa_1/\kappa_2 \quad \Rightarrow \quad \alpha = \pi/2 \Leftrightarrow \kappa_1 + \kappa_2 = 0$$

Applications: Minimal Surfaces

Total Energy

$E = E_{Ortho.} + E_{Anet} + \omega_1 E_{fair}$



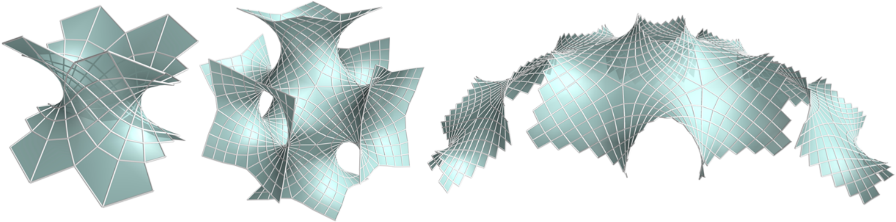
Energy term for A-nets

$$E_{Anet} = \sum_{i=1}^{|V|} \sum_{j=1}^{4} (n_i \cdot (v_{ij} - v_i))^2$$
$$+ \sum_{i=1}^{|V|} (n_i \cdot n_i - 1)^2$$

Applications: Minimal Surfaces

Total Energy

$E = E_{Ortho.} + E_{Anet} + \omega_1 E_{fair}$



Applications: Asymptotic Gridshell