Optimization Techniques for Geometry Processing

Justin Solomon Sebastian Claici

Inequality constrained optimization

General form

objective function

$f : \mathbb{R}^n \to \mathbb{R}$

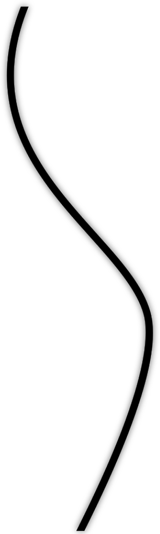
$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & f(x) \\
\text{subject to} \quad & g_i(x) \leq 0, \; i = 1, \ldots, m.
\end{aligned}
$$

$g_i : \mathbb{R}^n \to \mathbb{R}$

constraints

Geometric interpretation

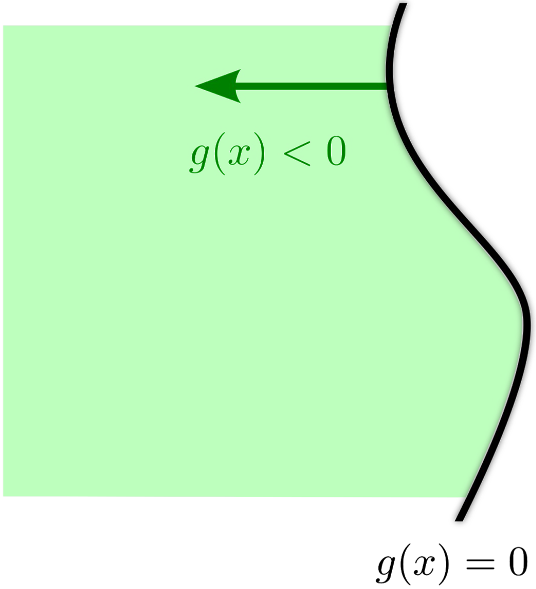
$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \le 0, \; i = 1, \dots, m. \end{aligned}$$



$g(x) = 0$

Geometric interpretation

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & g_i(x) \leq 0, \; i = 1, \ldots, m. \end{array}$$



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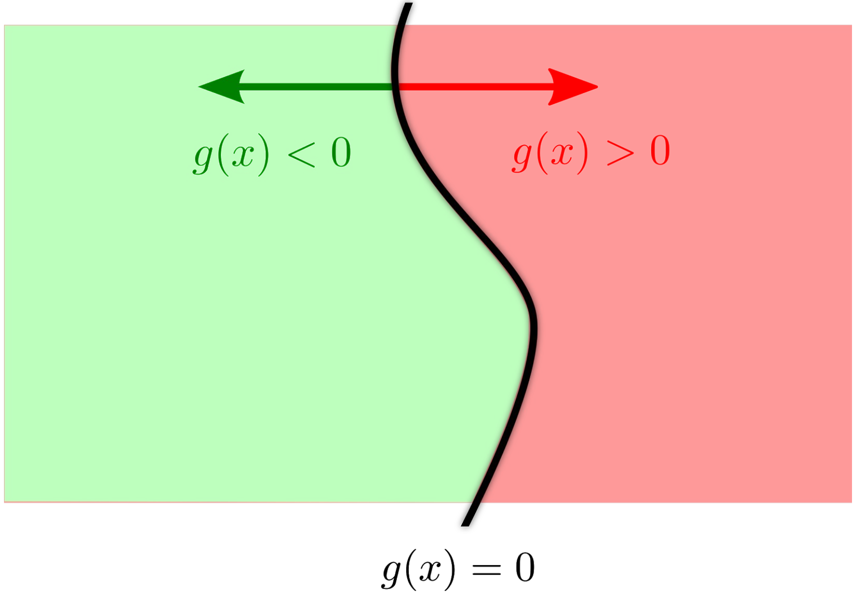

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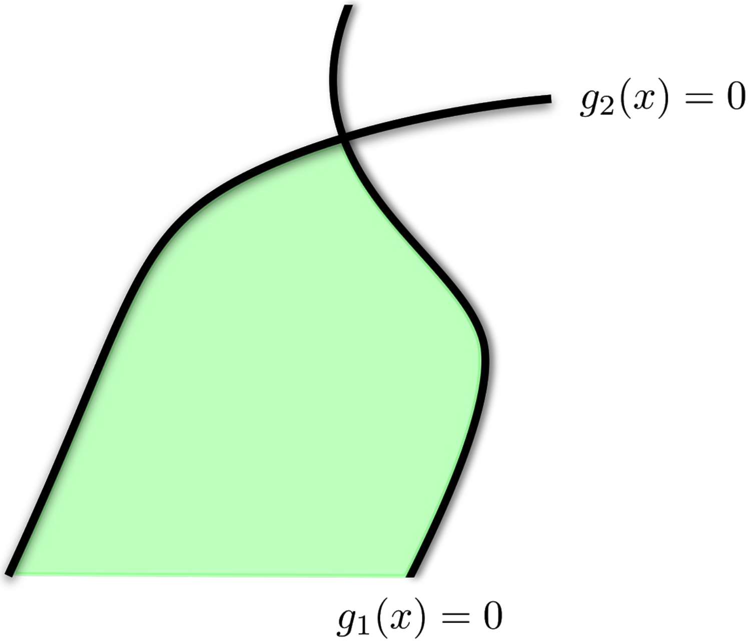

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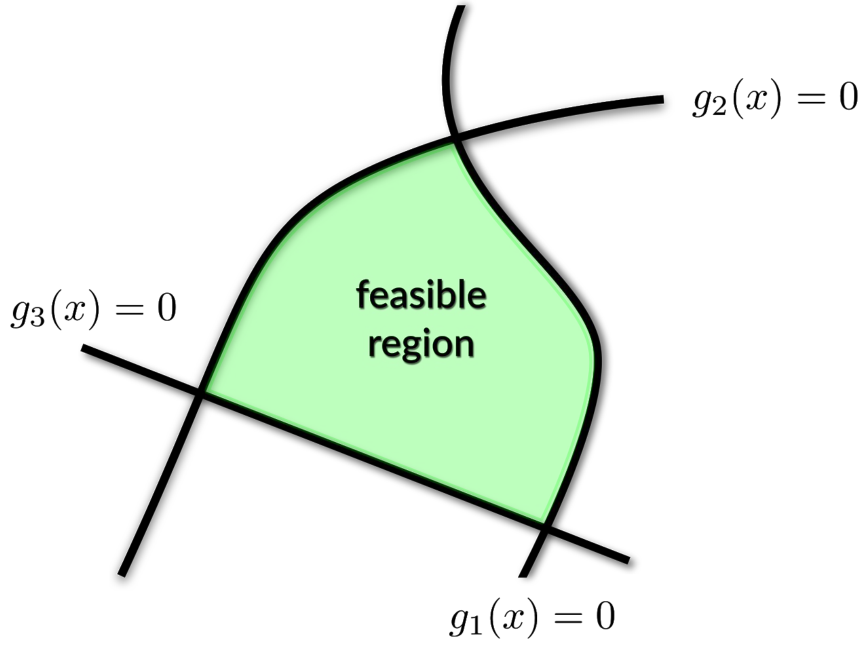
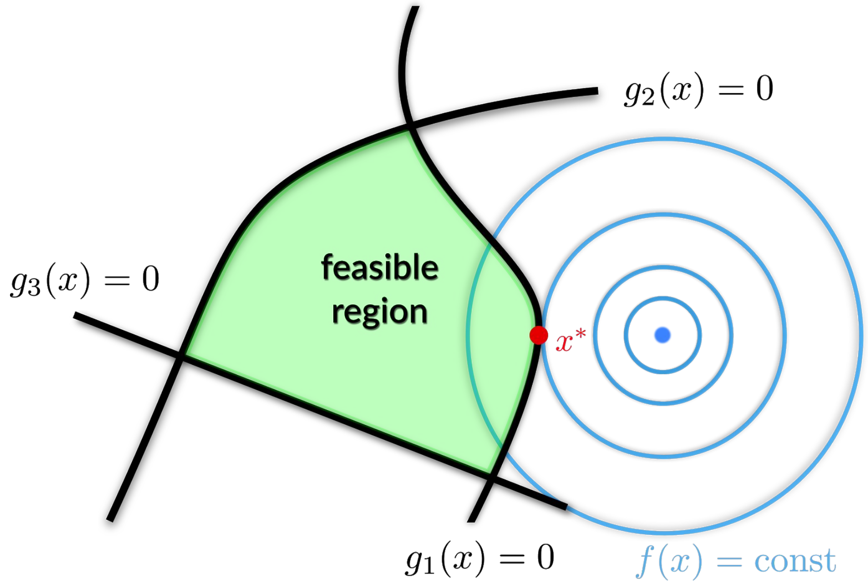
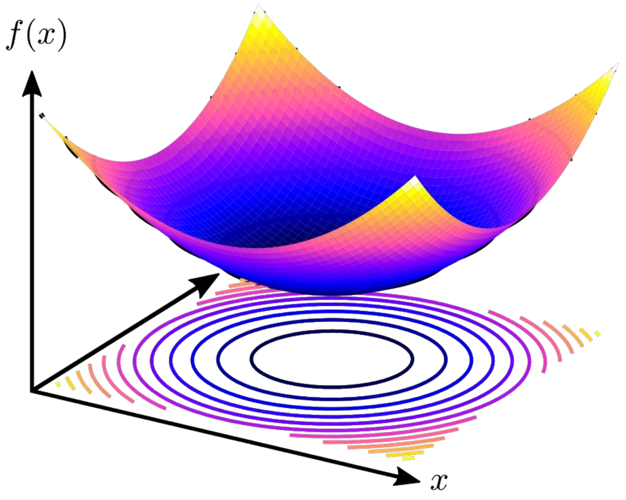


Geometric interpretation

$$\underset{x}{\text{minimize}} \quad f(x)$$
$$\text{subject to} \quad g_i(x) \le 0, \; i = 1, \dots, m.$$





Convexity

Searching for globally optimal solutions *usually* requires convexity.

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$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$$

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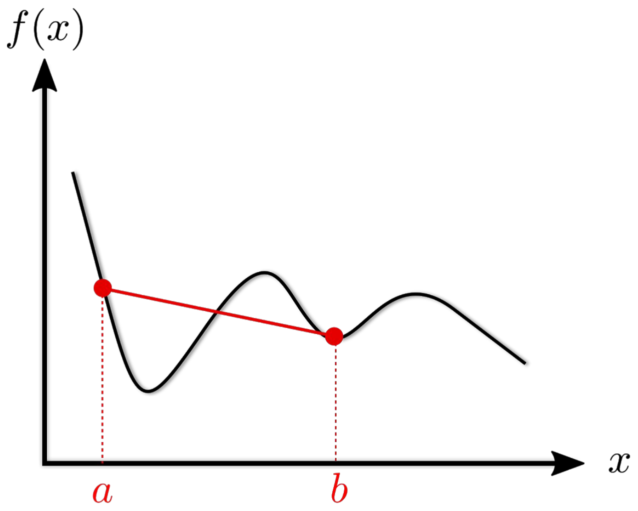
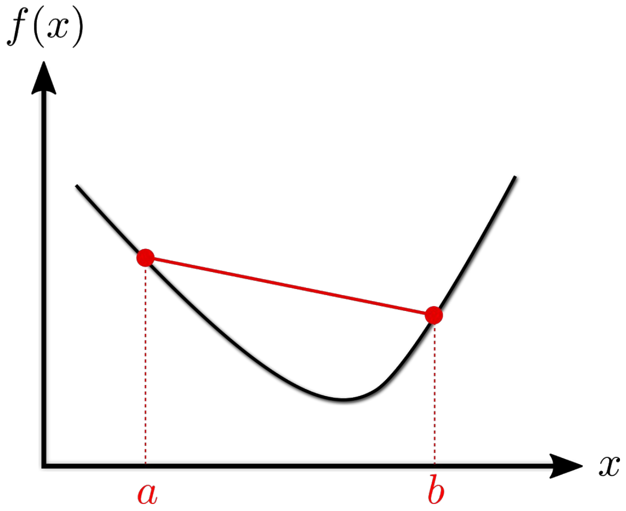


non-convex

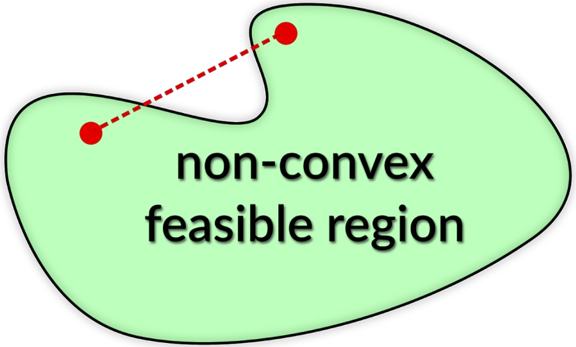


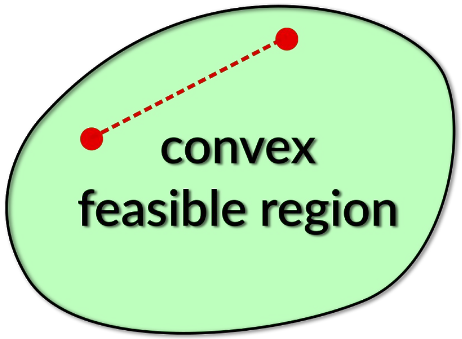
convex

Convexity

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0,\ i = 1, \ldots, m. \end{aligned}$$

is a **convex optimization problem** if $f(x)$ and $g_i(x)$ are convex functions





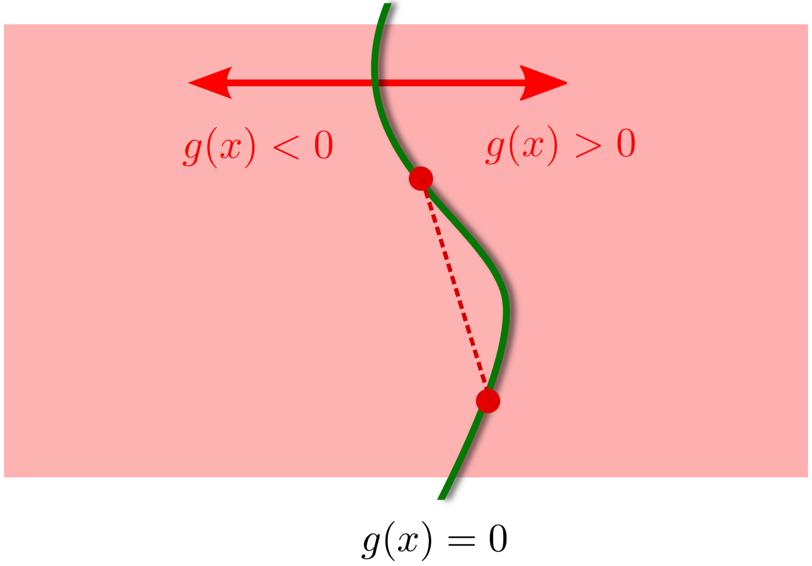
Convexity

Equality constraints must be **affine**.

Equivalent:

$$g(x) = 0$$

$$g(x) \leq 0 \text{ and } g(x) \geq 0$$



Problems with convex objectives and convex constraints

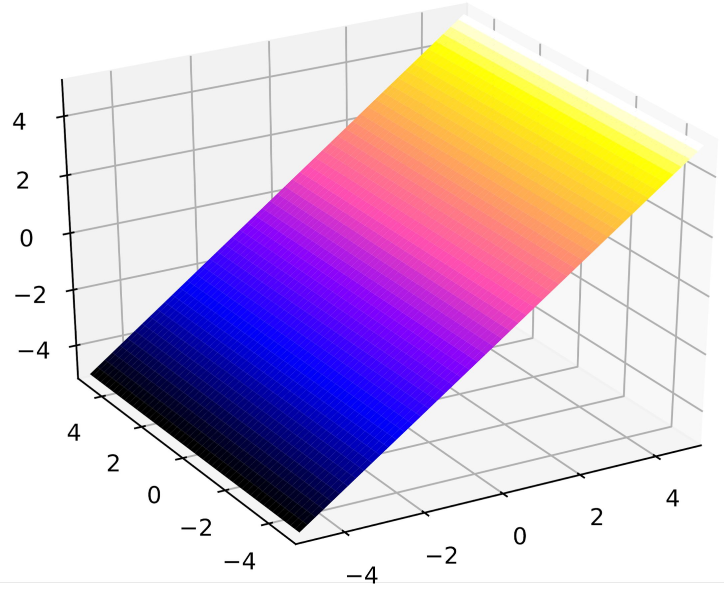
can be solved to global optimality.

Convex problem classes

Linear program (LP)

General form

$$\begin{array}{ll} \underset{x}{\text{minimize}} & c^\top x \\ \text{subject to} & Ax \leq b \end{array}$$



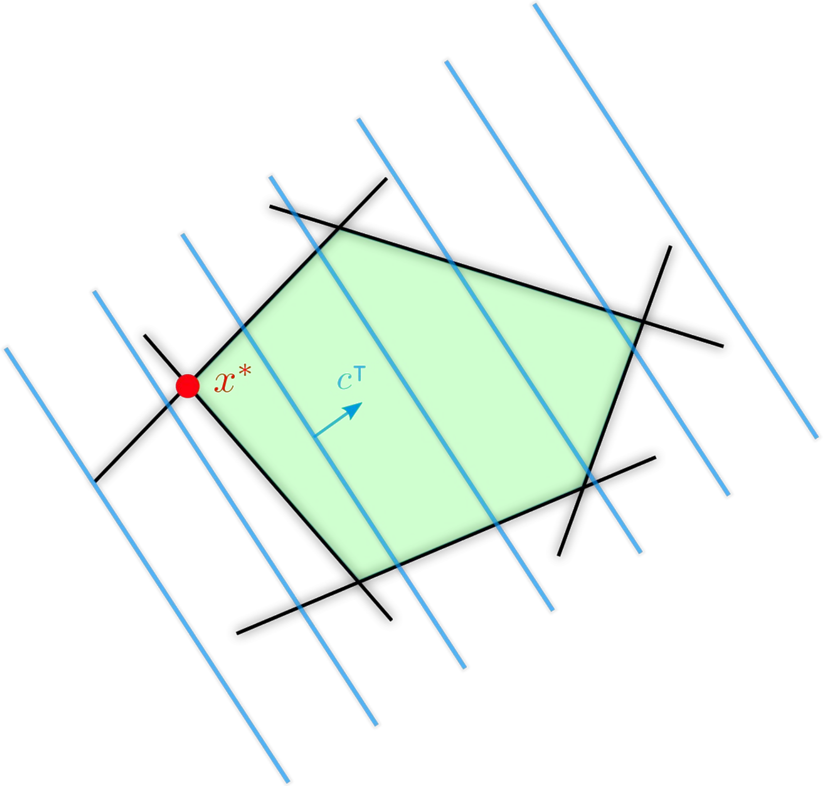
Linear program (LP)

General form

linear objective

$$\underset{x}{\text{minimize}} \quad c^{\top} x$$
$$\text{subject to} \quad Ax \leq b$$

linear constraints
i.e.
each constraint is a half space
i.e.
feasible set is a polyhedron



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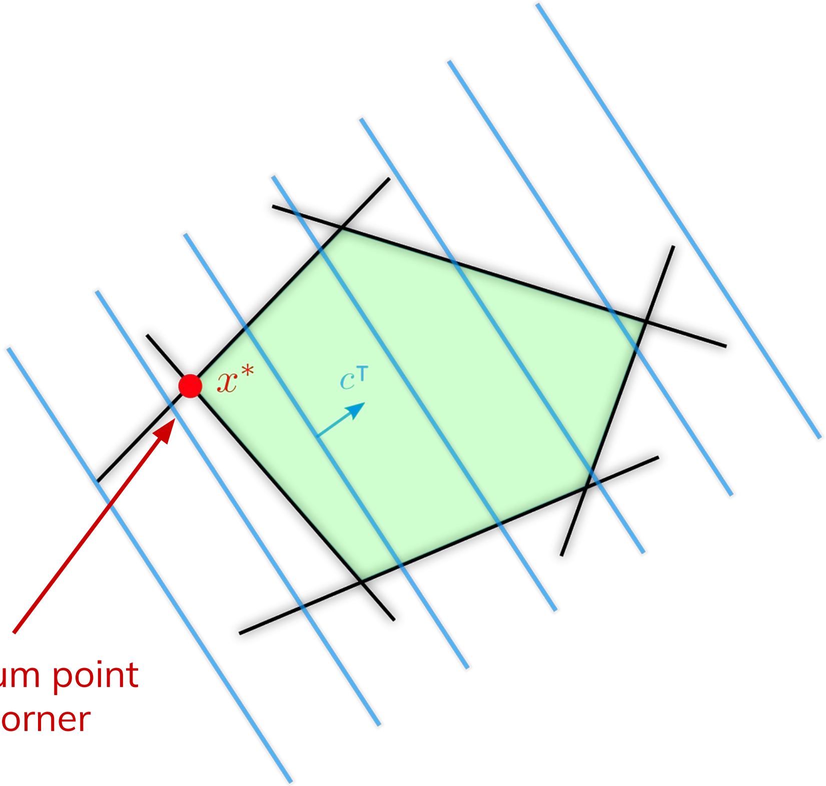
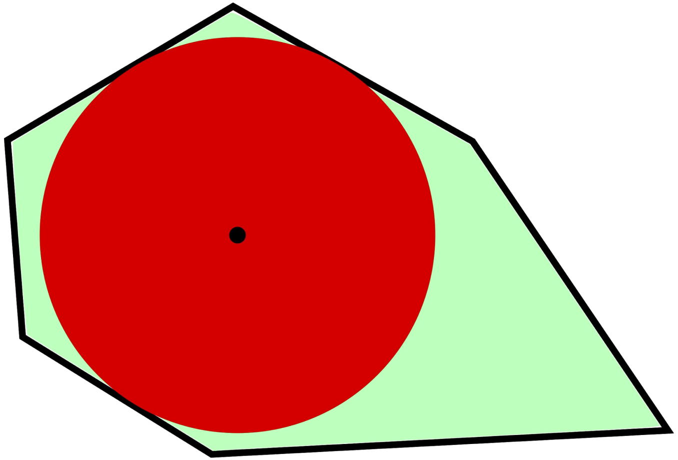


optimum point
at corner

Example LP

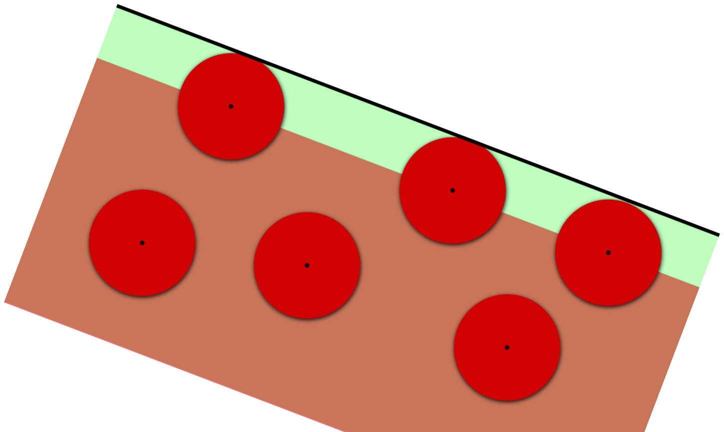
Find the largest ball enclosed in a polyhedron.



Ball $\mathcal{B} = \{x_c + u \mid \|u\|_2 \leq r\}$

Ball of radius r is inside halfspace $a^\intercal x \leq b$ if

$a^\intercal x + r\|a\|_2 \leq b$



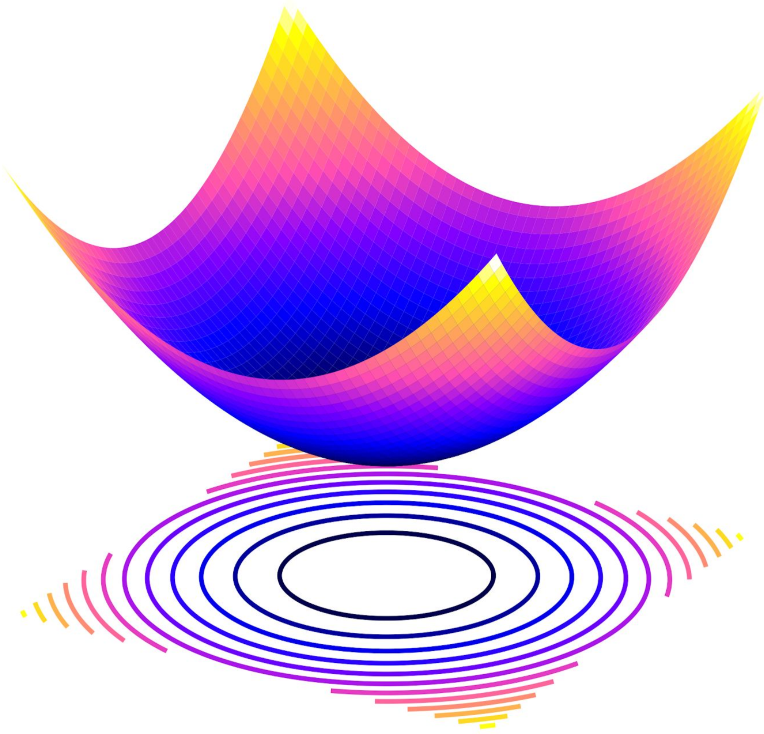
$$\begin{array}{ll} \underset{r,x}{\text{minimize}} & -r \\ \text{subject to} & a_i^\intercal x + r\|a_i\|_2 \leq b_i, \ i = 1, \ldots, m. \end{array}$$

Quadratic program (QP)

General form

quadratic objective

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \frac{1}{2}x^\intercal Q x + p^\intercal x + c \\ \text{subject to} \quad & Ax \leq b \end{aligned}$$



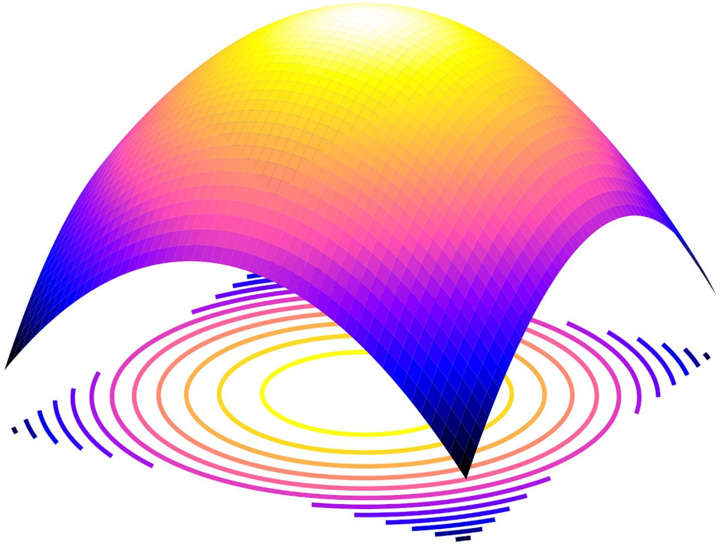
Positive definite Q

Quadratic program (QP)

General form

quadratic objective

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \frac{1}{2}x^\intercal Q x + p^\intercal x + c \\ \text{subject to} & Ax \leq b \end{array}$$



Negative definite Q

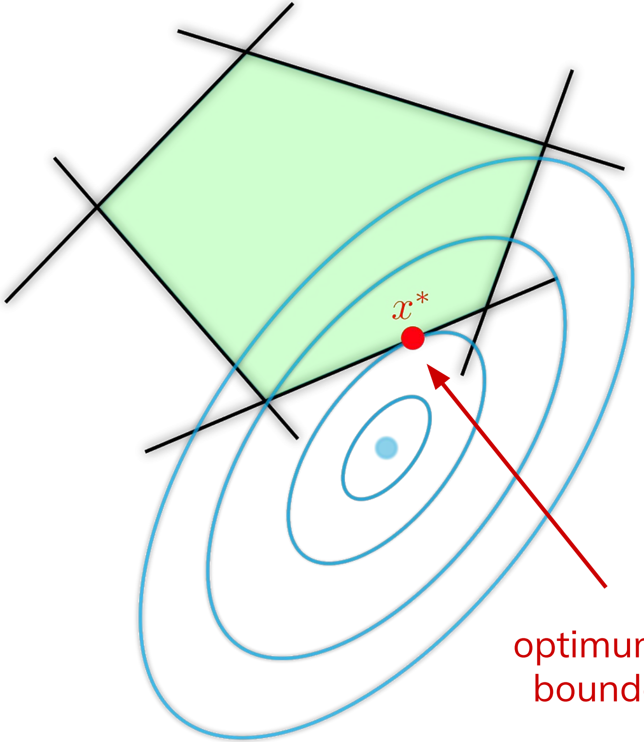
Quadratic program (QP)

General form

quadratic objective

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \frac{1}{2} x^\intercal Q x + p^\intercal x + c \\ \text{subject to} \quad & Ax \leq b \end{aligned}$$

linear constraints
i.e.
each constraint is a half space
i.e.
feasible set is a polyhedron



optimum on boundary

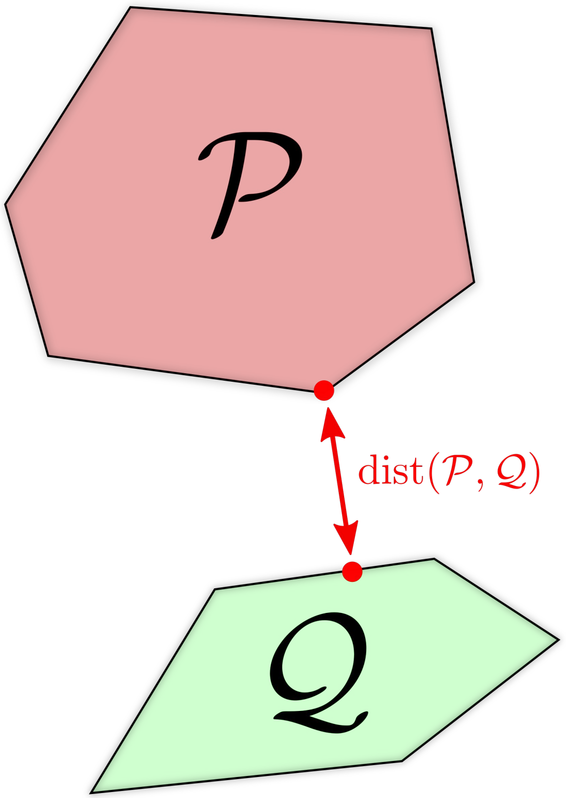
Example QP

Distance between two polyhedra:

$$\mathrm{dist}(\mathcal{P}, \mathcal{Q}) = \inf\{\|p - q\|_2 \mid p \in \mathcal{P}, q \in \mathcal{Q}\}$$

Quadratic program

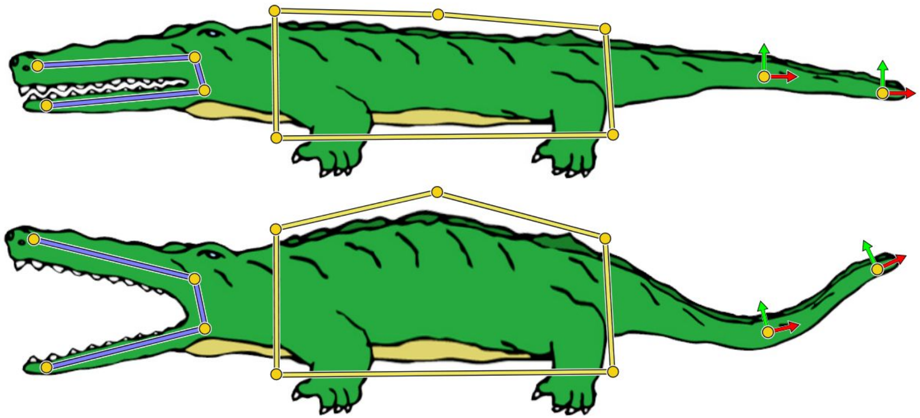
$$\begin{array}{ll} \underset{x=(p,q)}{\text{minimize}} & \|p - q\|_2^2 \\ \text{subject to} & A_{\mathcal{P}} p \leq b_{\mathcal{P}} \\ & A_{\mathcal{Q}} q \leq b_{\mathcal{Q}} \end{array}$$



Example QP

Real time object deformation [**Jacobson et al., 2011**]

$$\begin{aligned}
\underset{w_j,\ j=1,\dots,m}{\arg\min} \quad & \sum_{j=1}^{m} \frac{1}{2} \int_{\Omega} \|\Delta w_j\|^2 dV \\
\text{subject to:} \quad & w_j|_{H_k} = \delta_{jk} \\
& w_j|_F \ \text{is linear} \\
& \sum_{j=1}^{m} w_j(\mathbf{p}) = 1 \\
& 0 \le w_j(\mathbf{p}) \le 1, \ \ j = 1, \dots, m,
\end{aligned}$$



Second order cone program (SOCP)

General form

linear objective

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & a^{\intercal} x \\
\text{subject to} \quad & \|A_i x + b_i\|_2 \leq c_i^{\intercal} x + d_i, \; i = 1, \ldots, m.
\end{aligned}
$$

Second order cone program (SOCP)

General form

linear objective

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & a^\intercal x \\
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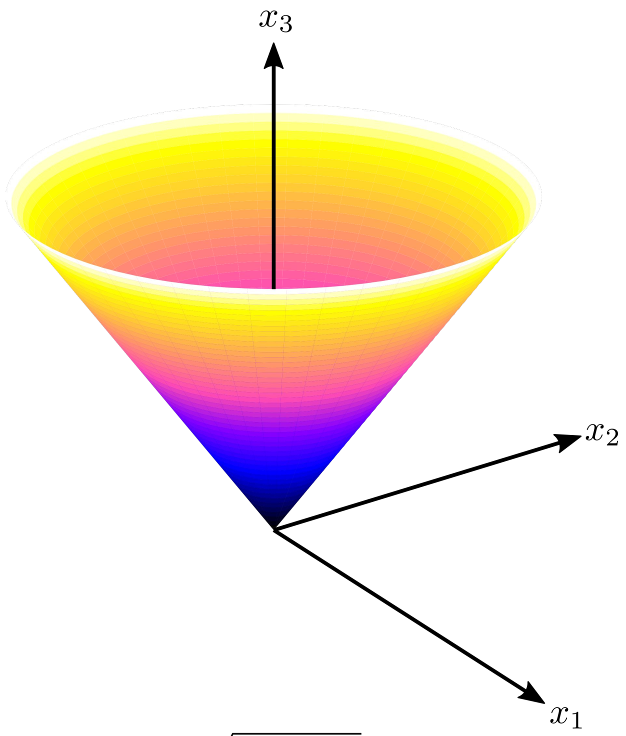
\end{aligned}
$$

cone constraint



$$\sqrt{x_1^2 + x_3^2} \leq x_2$$

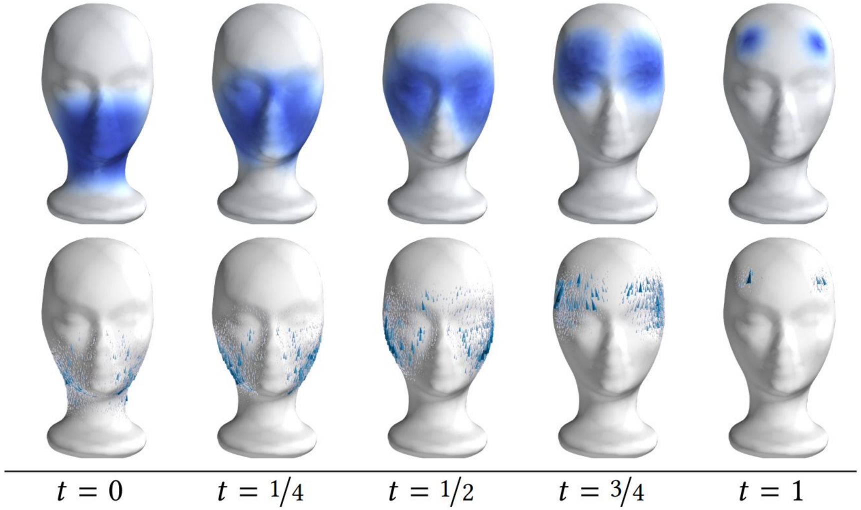
Example SOCP

Dynamical optimal transport [**Lavenant et al. 2018**]

linear objective

$$\underset{\varphi}{\text{minimize}} \quad \int_{\mathcal{M}} \varphi^0 \mathrm{d}\mu^0 - \int_{\mathcal{M}} \varphi^1 \mathrm{d}\mu^1$$
$$\text{subject to} \quad \partial_t \varphi + \frac{1}{2}\|\nabla\varphi\|_2^2 \leq 0, \text{ on } [0,1] \times \mathcal{M}.$$

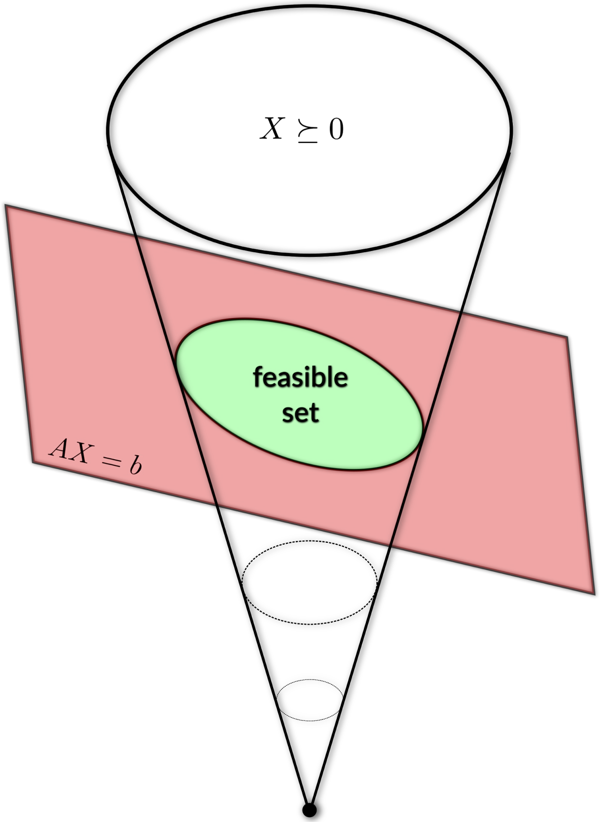
cone constraint



Semidefinite program (SDP)

General form

$$
\begin{array}{ll}
\underset{X}{\text{minimize}} & \operatorname{tr}(CX) = \sum_{ij} C_{ij} X_{ij} \\
\text{subject to} & \operatorname{tr}(A_i X) \leq b_i,\ i = 1, \ldots, m \\
& X \succeq 0
\end{array}
$$



Example SDP

Surface correspondence [**Maron et al. 2016**].

$$\min_{X,R} \|RP - QX\|_F^2$$

$$X\mathbf{1} = \mathbf{1}, \quad \mathbf{1}^T X = \mathbf{1}^T$$

$$X_j X_j^T = \mathrm{diag}(X_j), \quad j = 1 \ldots n$$

$$RR^T = R^T R = I$$

Can relax final two constraints:

$$Z_j \succeq \begin{bmatrix} X_j \\ [R] \end{bmatrix} \begin{bmatrix} X_j \\ [R] \end{bmatrix}^T$$

Hierarchy of convex programs

$$\text{LP} \subset \text{QP} \subset \text{QCQP} \subset \text{SOCP} \subset \text{SDP}$$

Algorithms

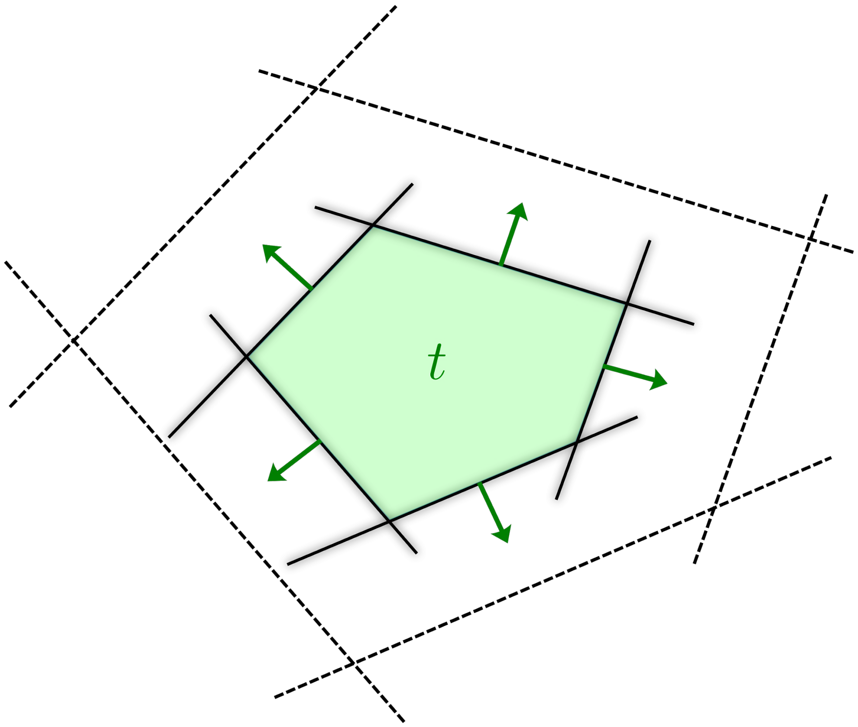
Finding a feasible point

Start with:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0, \; i = 1, \ldots, m. \end{aligned}$$

Replace with a **feasibility** problem

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & t \\ \text{subject to} \quad & g_i(x) \leq t, \; i = 1, \ldots, m. \end{aligned}$$



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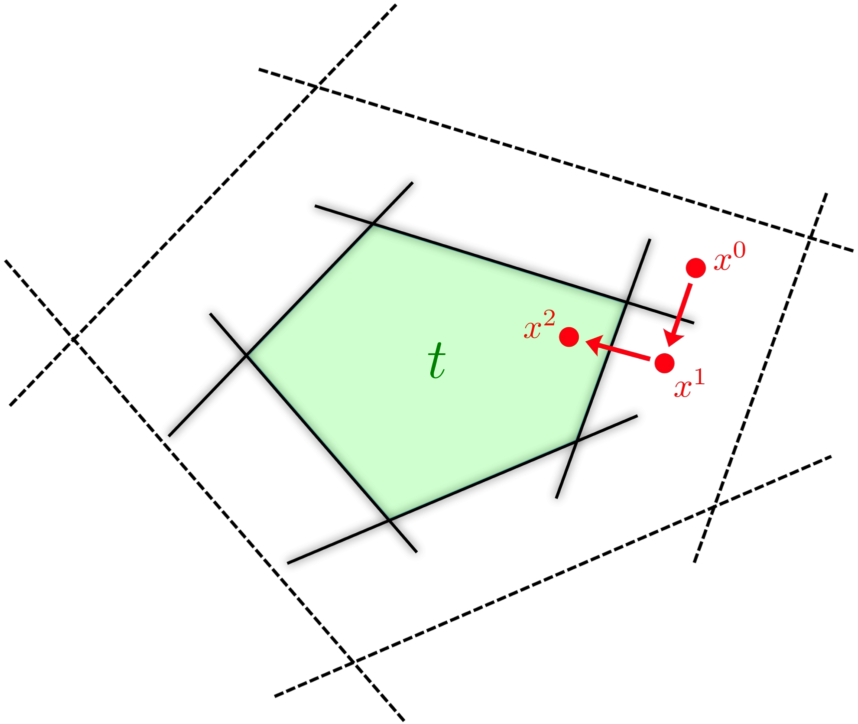
$$\begin{aligned} \underset{x}{\text{minimize}} \quad & t \\ \text{subject to} \quad & g_i(x) \le t, \; i = 1, \ldots, m. \end{aligned}$$

Stop when t ≤ 0



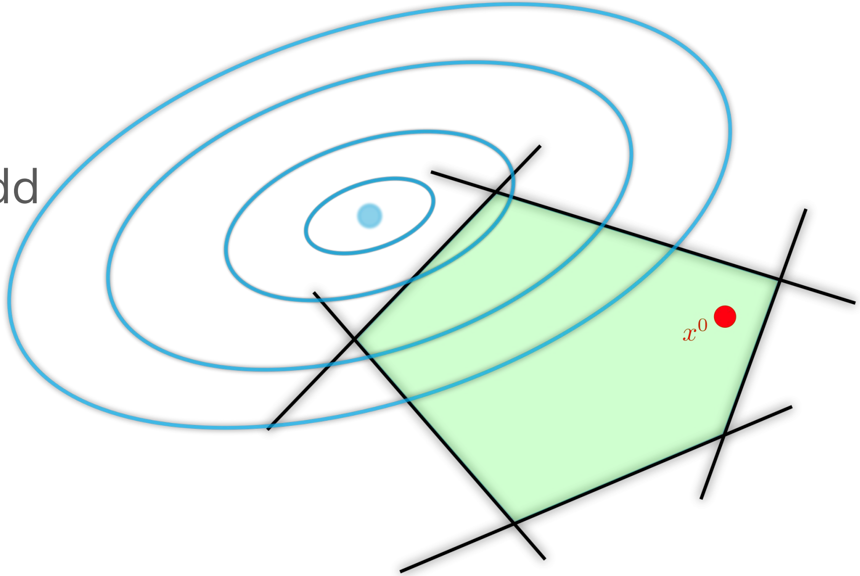
Active set methods

Active set

Treat problem as equality constrained.

Maintain an active constraint set *A*:

- If an inequality constraint is violated add it as an equality constraint to A.
- Remove constraints that aren't active.



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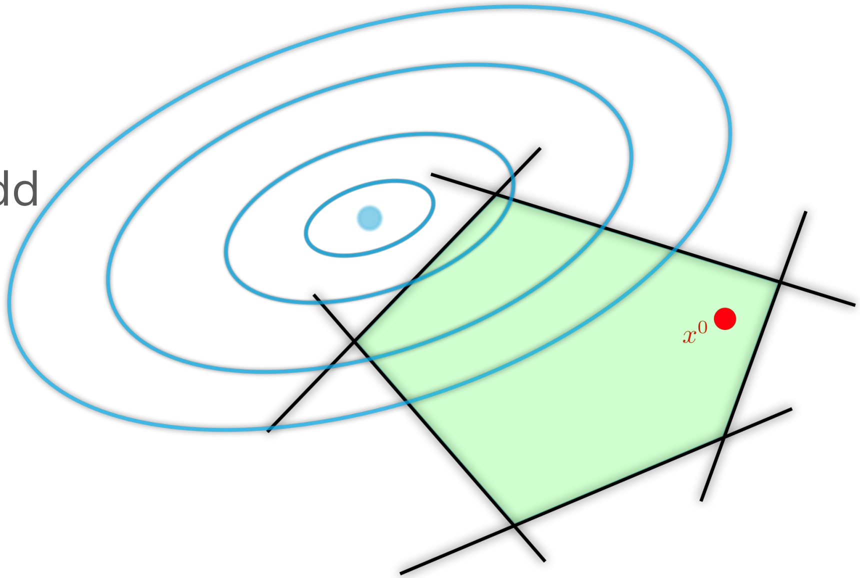
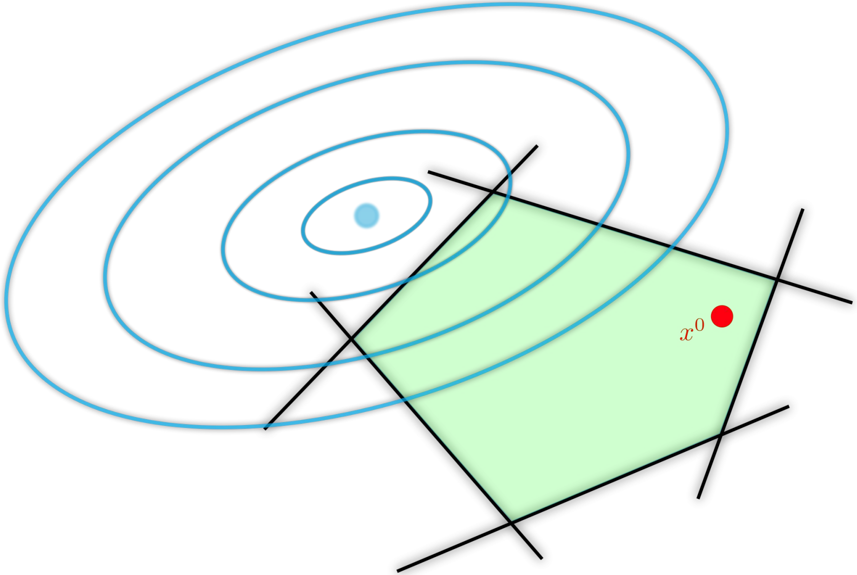


Recall: the **optimum** is on the boundary.

Active set

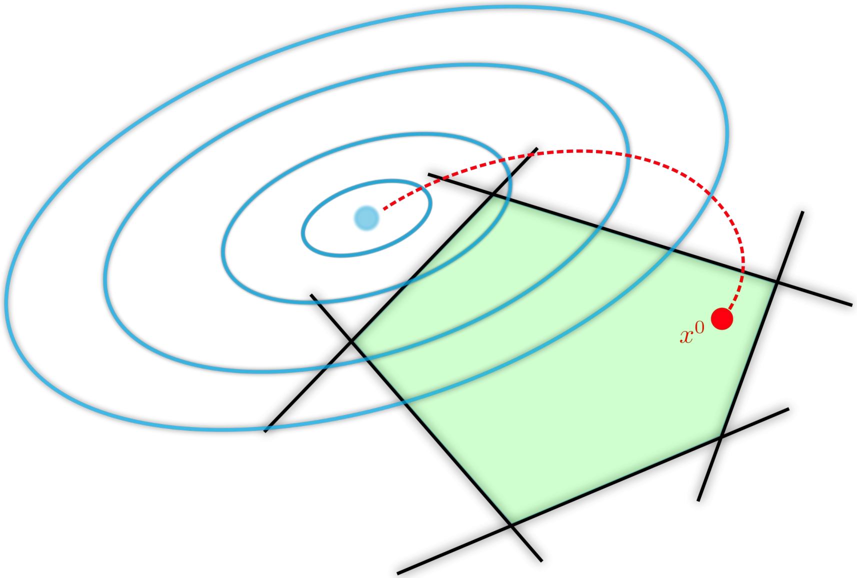
Start with feasible x^0.



Active set

Start with feasible x^0.

Find minimizer from x^0.



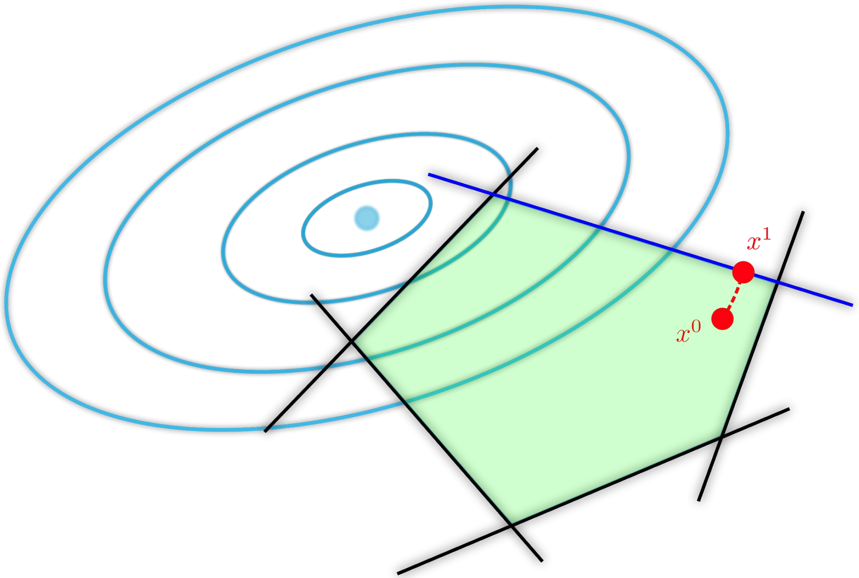
Active set

Start with feasible x^0.

Find minimizer from x^0.

Inequality constraint is violated:

- Add equality constraint to active set.



Active set

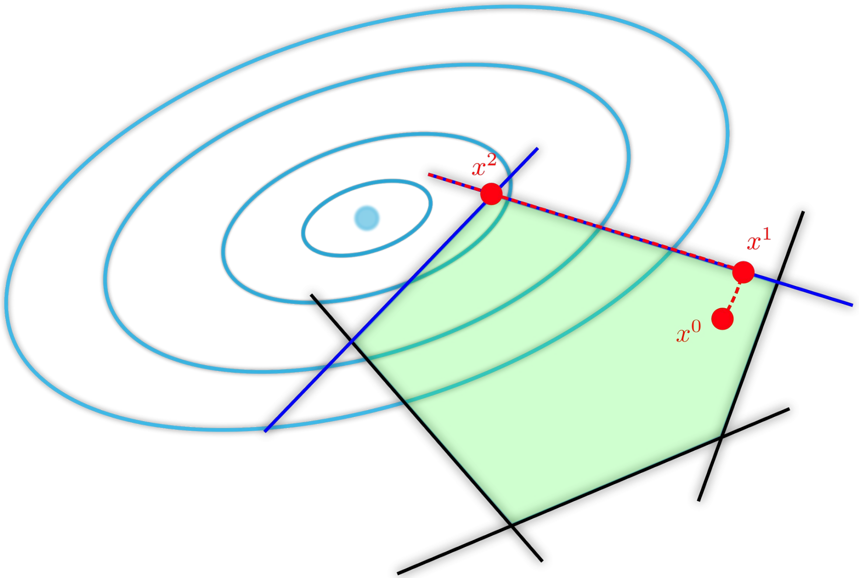
Start with feasible x^0.

Find minimizer from x^0.

Inequality constraint is violated:

- Add equality constraint to active set.

Repeat until optimality.



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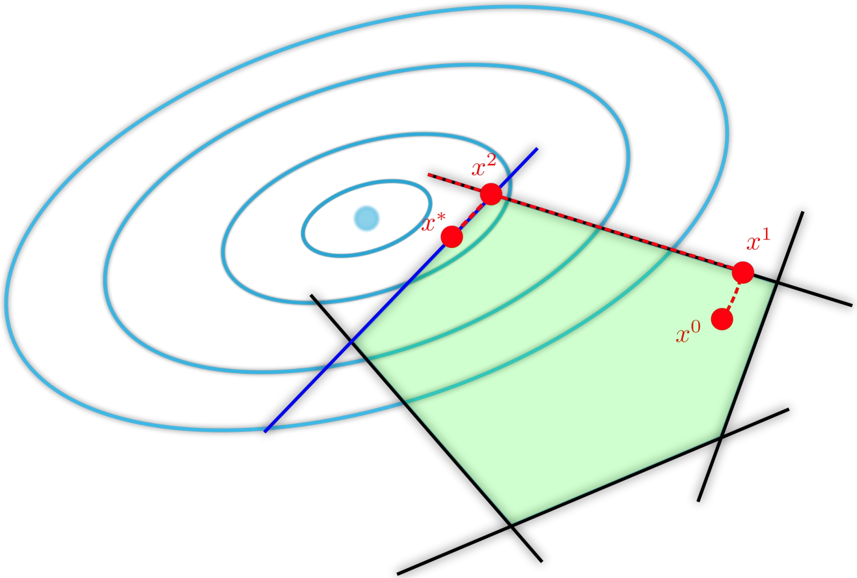


Active set method

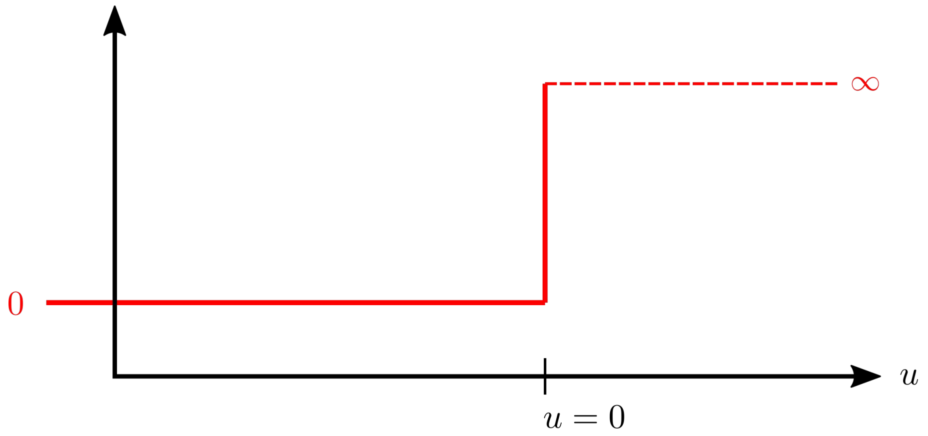
- Many variants
  - primal, dual, primal-dual methods, etc.
- Many specializations
  - LPs, QPs, ...
- Good choice for problems:
  - with linear and quadratic constraints
  - small or moderate amount of constraints
- Disadvantage
  - may require many iterations

Interior point methods

Interior point method

Replace constrained optimization by unconstrained problem.

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \le 0, \; i = 1, \ldots, m. \end{aligned}$$



Use the barrier function:

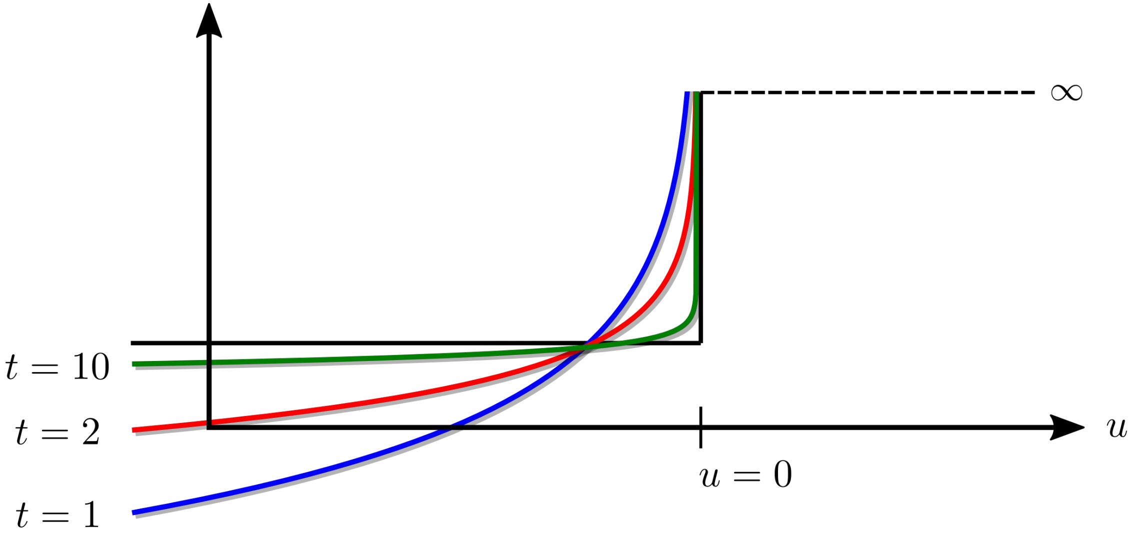
$$\underset{x}{\text{minimize}} \quad f(x) + \sum_{i=1}^{m} I_{-}(g_i(x))$$

$$I_{-}(u) = \begin{cases} 0 & \text{if } u \le 0 \\ +\infty & \text{if } u > 0 \end{cases}$$

Interior point method

Logarithmic barrier (smooth and convex approximation)

$$\hat{I}_-(u) = -\frac{1}{t}\log(-u)$$



	large t	small t
accuracy	high	low
speed	low	high

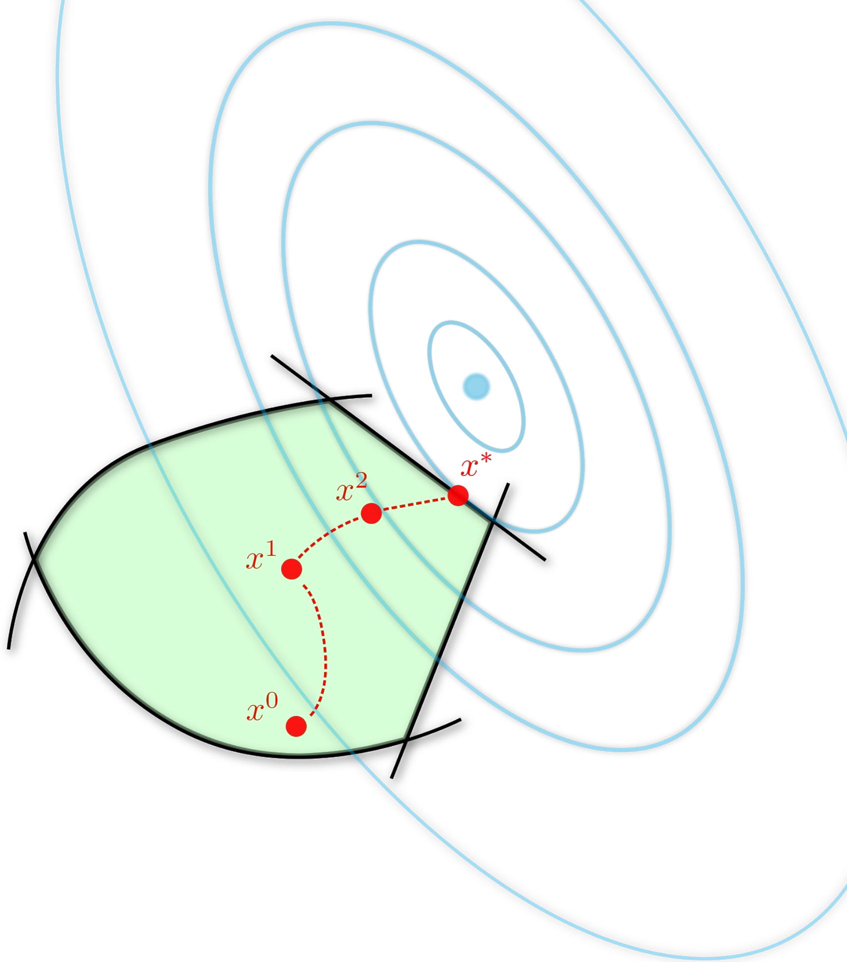
Interior point method

Solve

$$\underset{x}{\text{minimize}} \quad f(x) - \frac{1}{t}\sum_{i=1}^{m} \log(-g_i(x))$$

using unconstrained methods.

Set $t \leftarrow 10t$ and iterate until $\frac{m}{t} \leq \varepsilon$



Interior point method

- Many variants
  - primal-dual methods, reflective, etc.
- Many specializations
  - LPs, QPs, SDPs, ...
- Good choice for problems:
  - with many constraints
  - non-convex feasible regions
- Disadvantage
  - limited warm start capabilities

Summary

Optimization is everywhere in geometry processing.

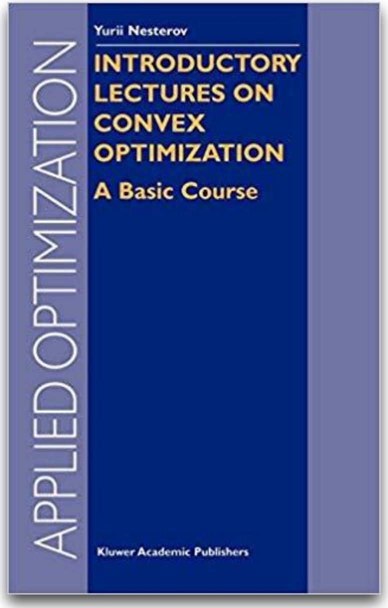
We've looked at algorithms for:

- Unconstrained optimization
- Equality constrained optimization
- Inequality constrained optimization

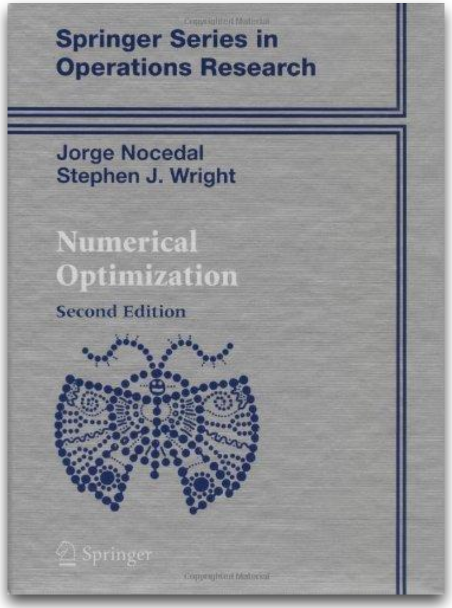
Take home messages:

- **Convexity** is crucial!
- Use the most **specialized** algorithm for best performance.

Further reading



Yurii Nesterov
Lectures on Convex Optimization
Springer, 2018



Jorge Nocedal and Stephen J. Wright
Numerical Optimization
Springer, 2006

Stephen Boyd and Lleven Vandenberghe
Convex Optimization
Cambridge University Press, 2004

Software

- Solvers:
  - Software from COIN-OR foundation
  - CPLEX
  - Gurobi
  - Mosek
- Interfaces
  - CVX
  - CoMISo
- Programming languages with good support
  - Python
  - MATLAB
  - Julia