# Nets in Discrete Differential Geometry

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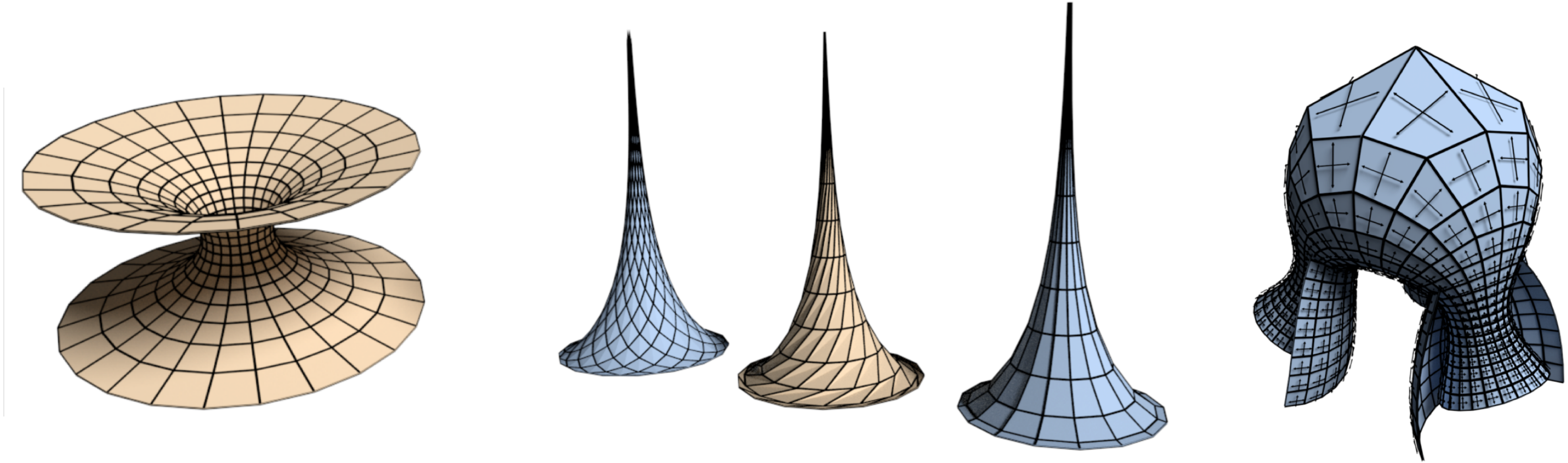
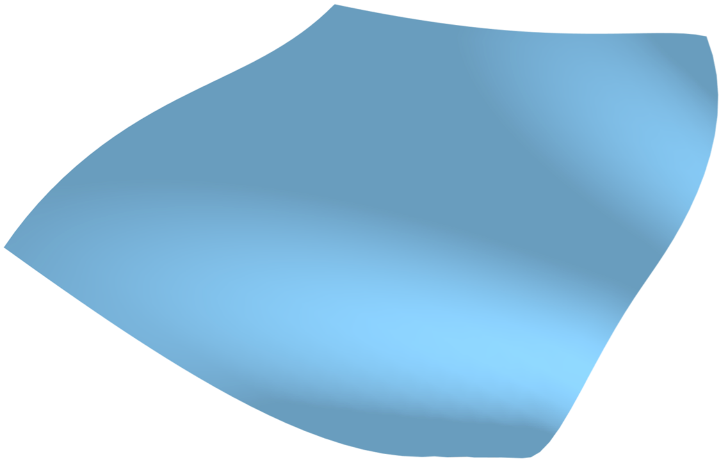
# What is this course about?

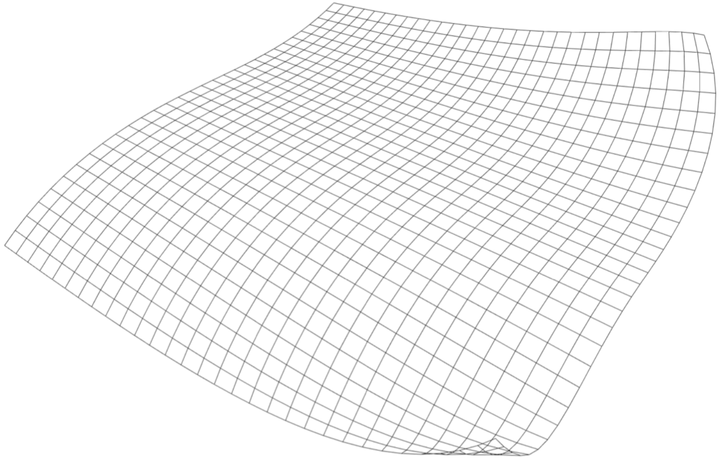
- In a nutshell: quad meshes that look like this:



[Hoffmann et al. 2014]

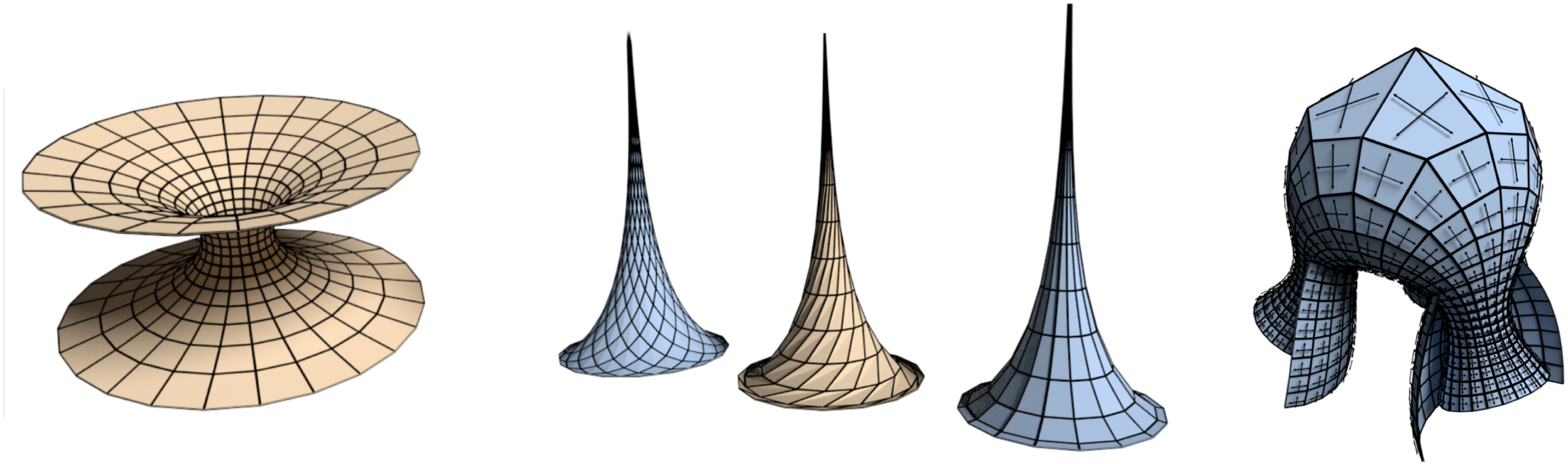
# What is this course about?





# What is this course about?

- Discretization of various geometries

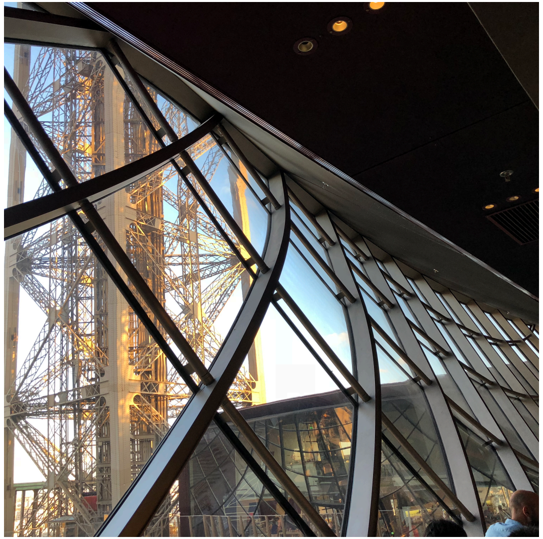


[Hoffmann et al. 2014]

# What is this course about?

- … but also like this:

Neumarkt17 furniture store in Zurich



Eiffel Tower pavilions, photo taken during **SGP 2018**


The British Museum, London

# Applications: Architecture

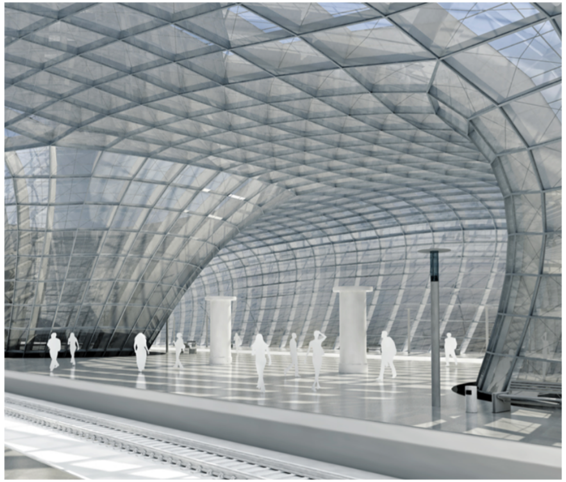
Approximation by glass panels

*Nur Alem*, Astana Kazakhstan

Save construction costs

Disney Concert Hall,
LA, Frank Gehry

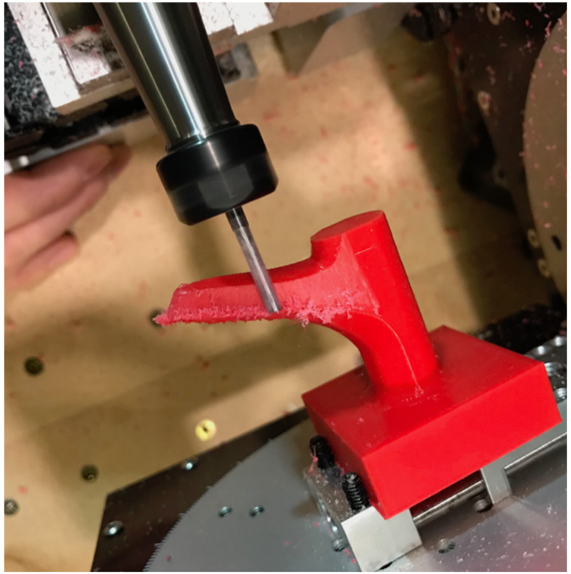
Support structures



[Liu et al. 2007]

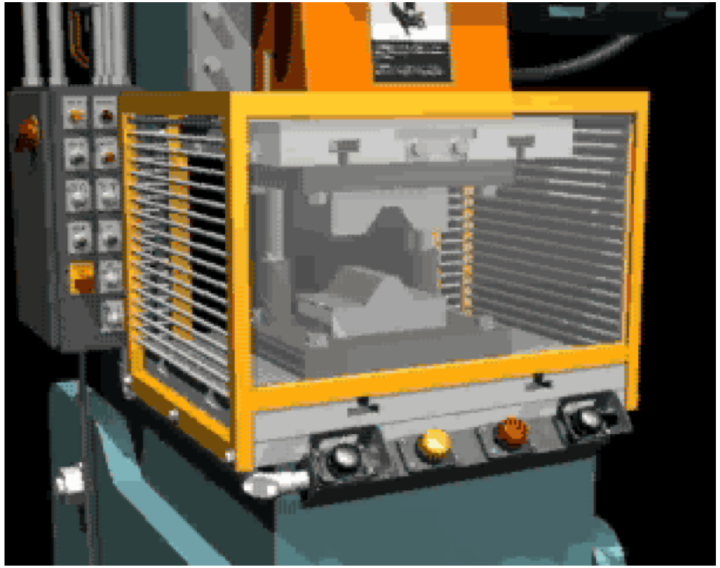
# Applications: Fabrication

CNC milling machines



[Stein et al. 2018]

Stamping (metalworking)



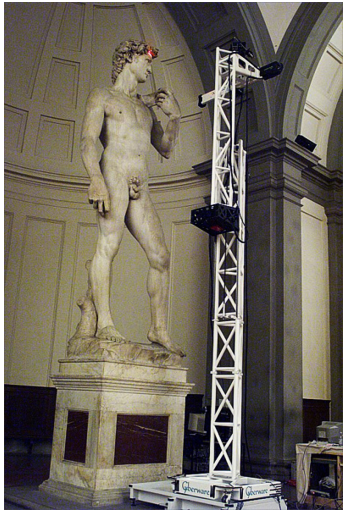
Stamping, wikipedia

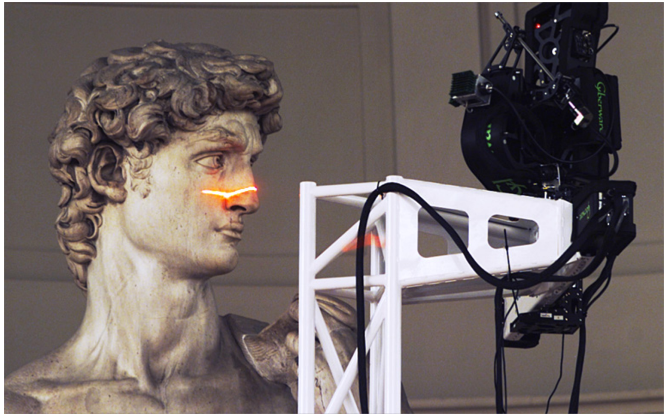
Wire mesh

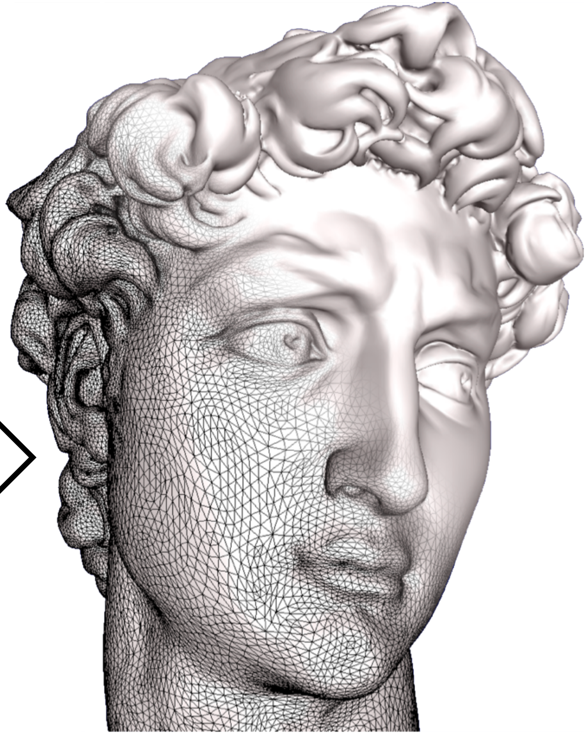
[Garg Akash et al. 2014]

# Not an arbitrary mesh

- Two main sources of meshes in geometry processing: **1) scanning**



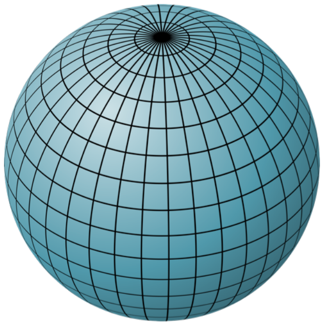


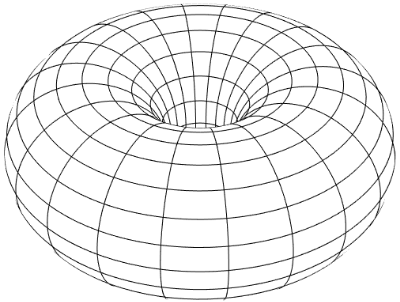


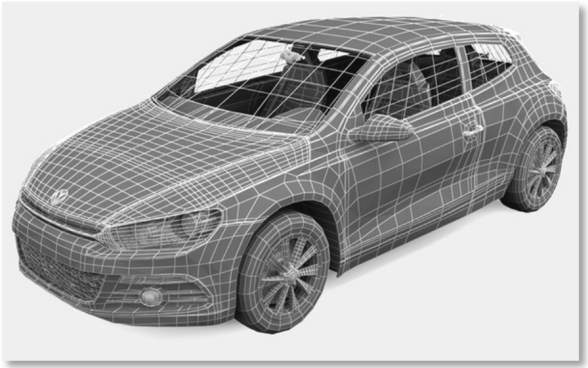
# Not an arbitrary mesh

- Two main sources of meshes in geometry processing: **2) sampling parametric surfaces**

$$(x, y, z) = f(u, v)$$





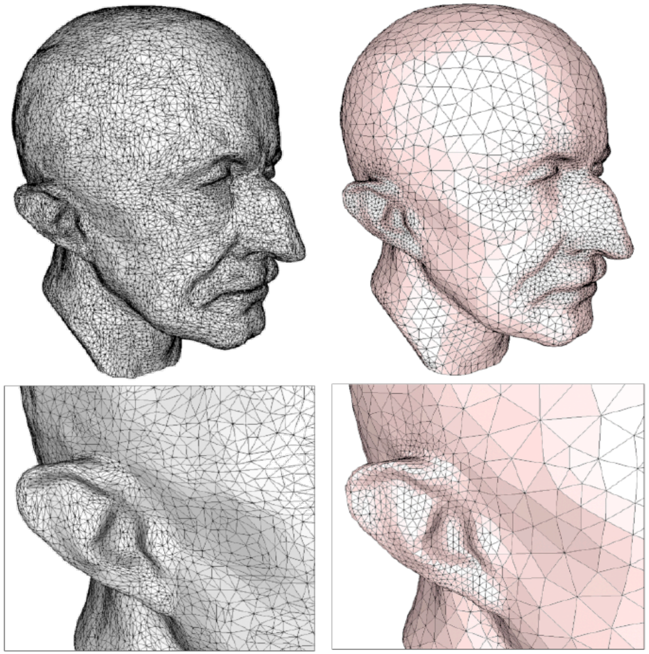


$$f(u, v) = \sum_{i,j} \mathbf{p}_{i,j} B_i(u) B_j(v)$$

# Meshes in geometry processing

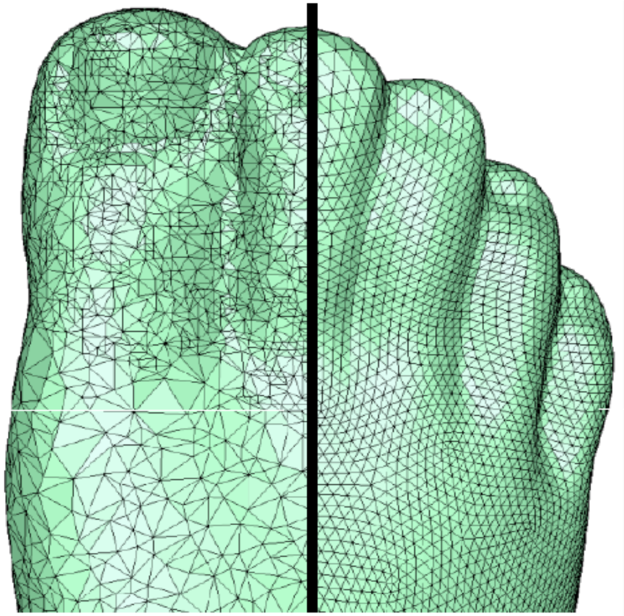
- Typically we care about the "underlying smooth (continuous) surface"

- Shape is king and the mesh is a necessary evil
  - Rendering
  - Solving geometric PDEs numerically using FEM



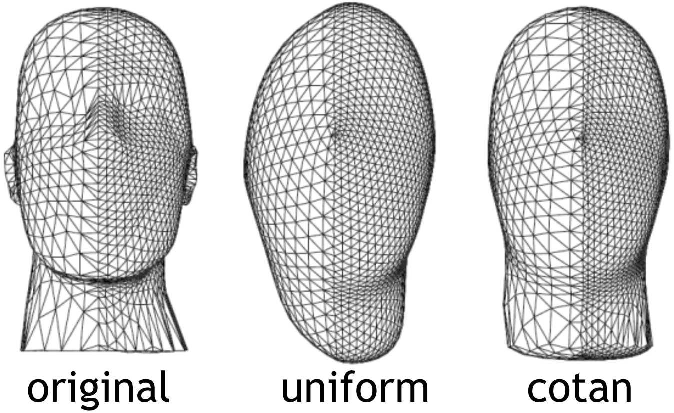
# Mesh independent algorithms

- Geometry processing algorithms are expected to be parameterization-agnostic
- Focus on shape, ignore the mesh
- **Mesh independence is key**



# Mesh independent algorithms

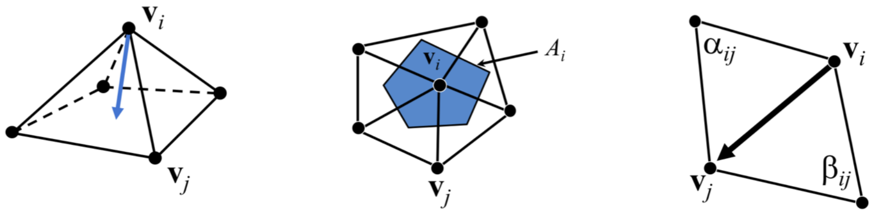
- Prime example: cotan Laplacian




[Desbrun et al. 1999]

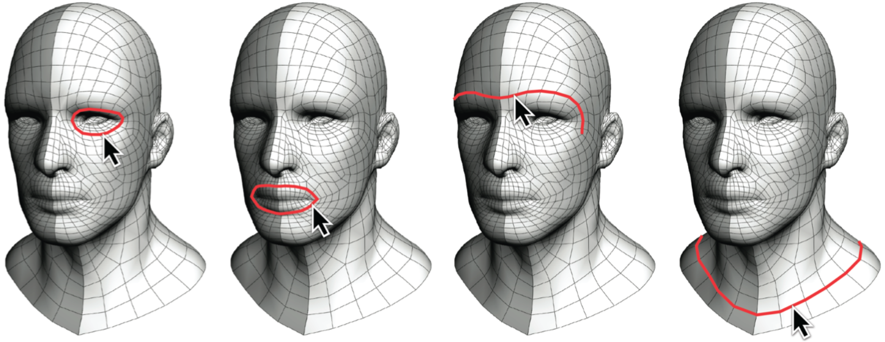
[Pinkall and Polthier 1993]

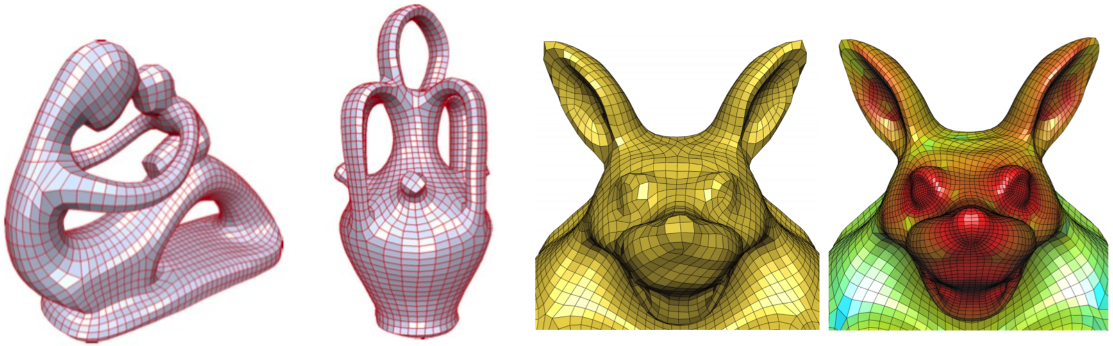
$$L_c(\mathbf{v}_i) = \frac{1}{A_i} \sum_{j \in \mathcal{N}(i)} \frac{1}{2} (\cot \alpha_{ij} + \cot \beta_{ij})(\mathbf{v}_j - \mathbf{v}_i)$$



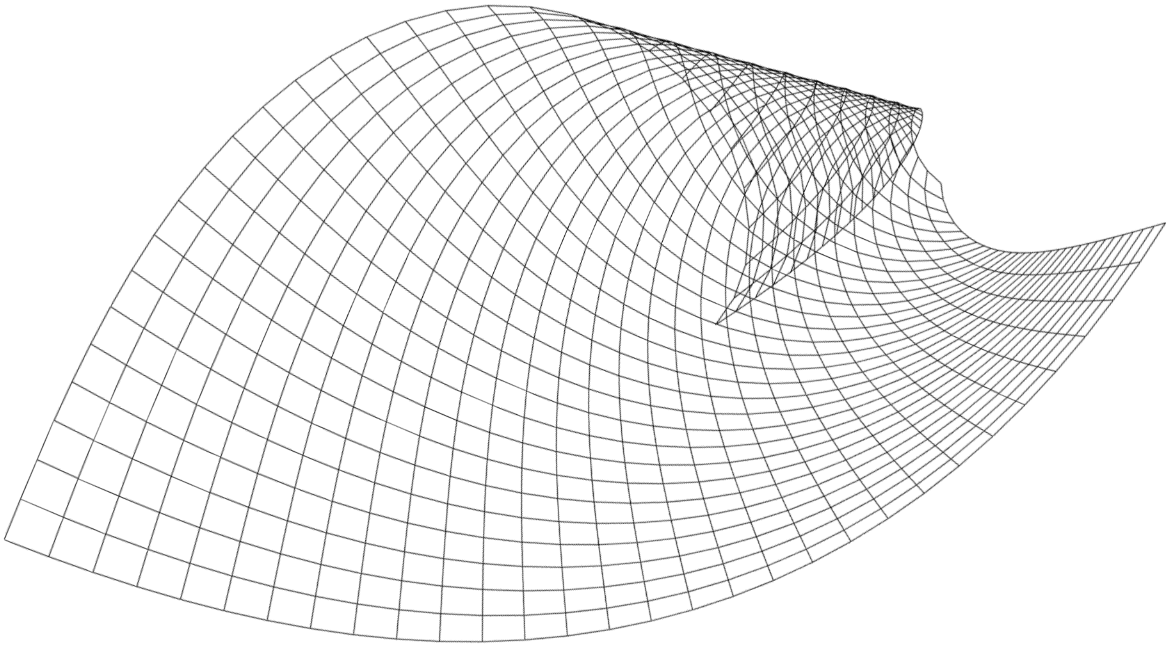
# Quad meshes in geometry processing

- Rising interest in research; widely used in industry (animators, architecture)
- Animators: intuitively align quad lines to semantic features, by hand
- In research: mostly curvature-aligned



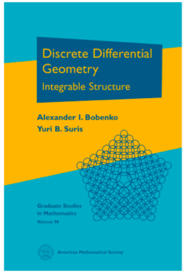


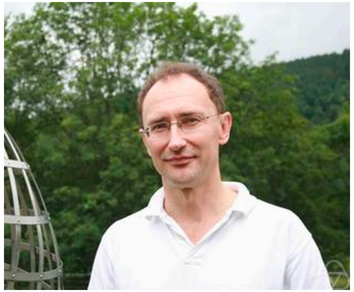
# No surface, only a mesh

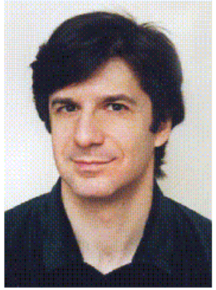


# Nets in DDG

- An area in pure mathematics
- Some of the key figures:





Alexander Bobenko


Yuri Suris


Ulrich Pinkall

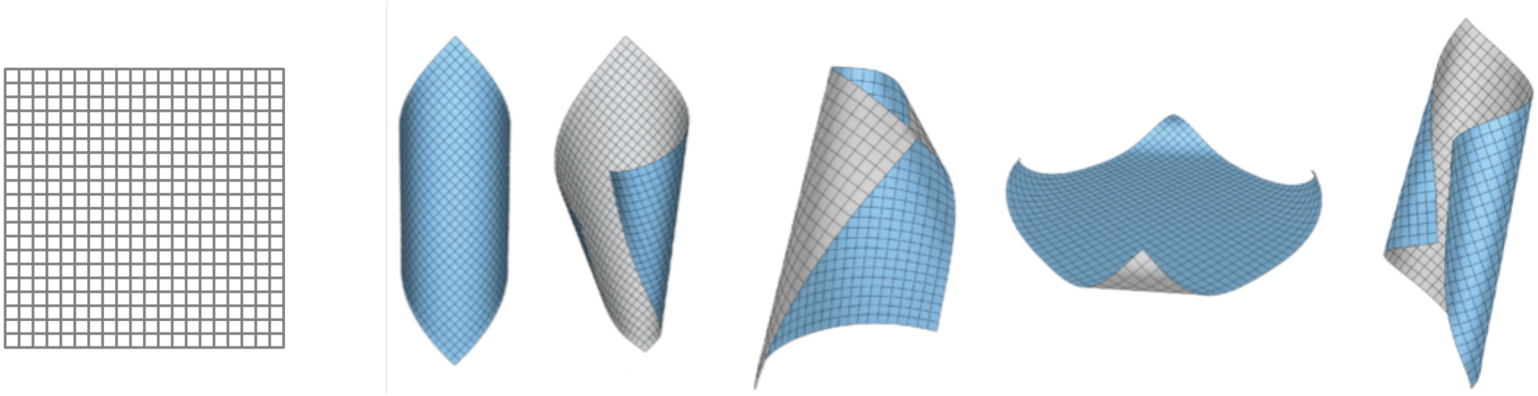

Helmut Pottmann


Tim Hoffmann

- **The Mesh is Everything**

# Nets in DDG

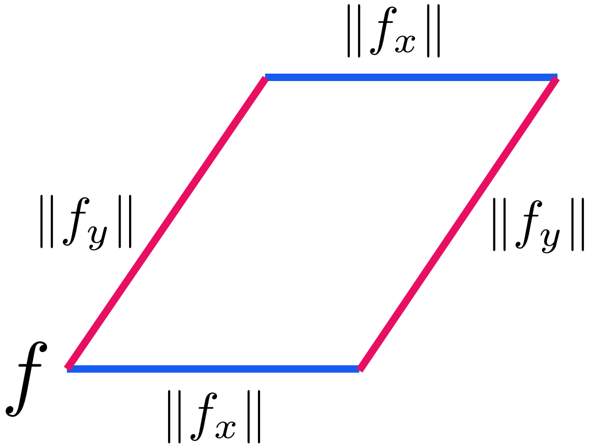
- Discrete analogs of **specific parameterizations**
- **Conditions** on the mesh to ensure it has the specific properties of a **given parameterization**



# Nets in DDG

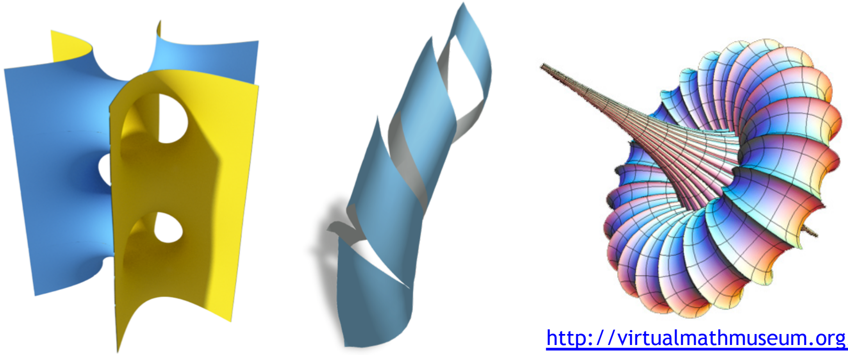
- Example: Chebyshev nets – parallelogram nets

[Wire Mesh Design
Garg, Akash, et al. 2014]



# Course Overview

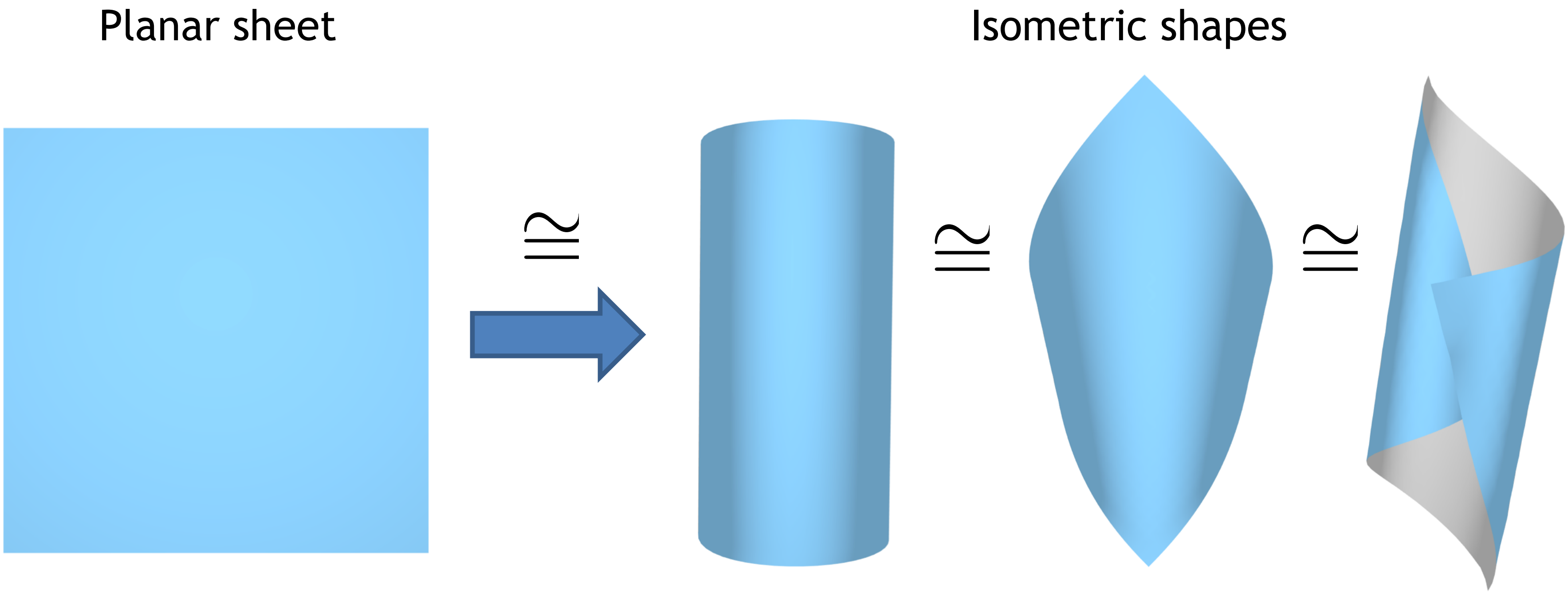
- The importance of choosing a mesh
- Background in Differential Geometry
- Discretizing various geometries
  - Applications



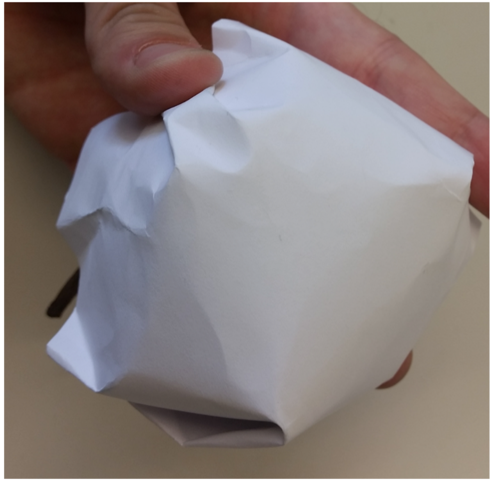
http://virtualmathmuseum.org/index.html

# The importance of a choice of mesh

# Bending paper or metal

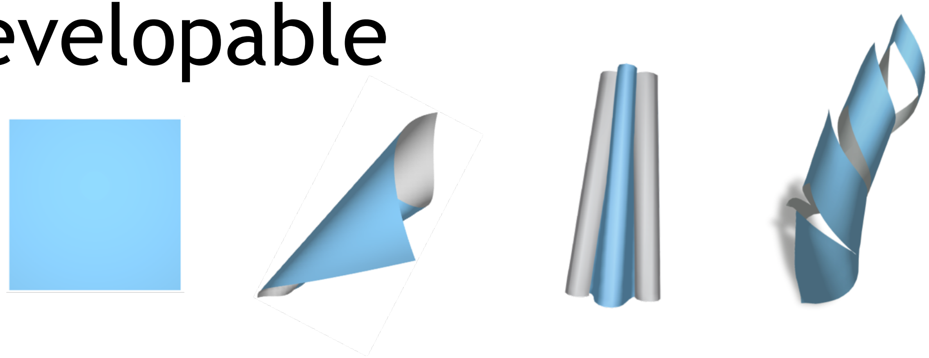


# Not developable



# Takeaway

- Many shapes are developable



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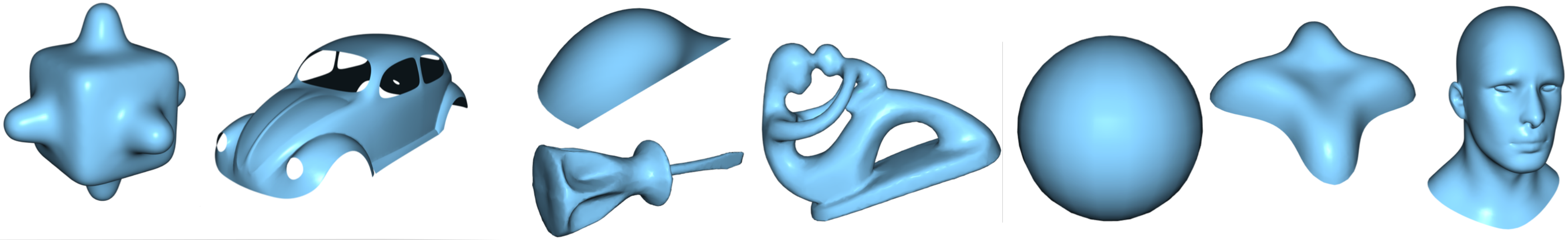
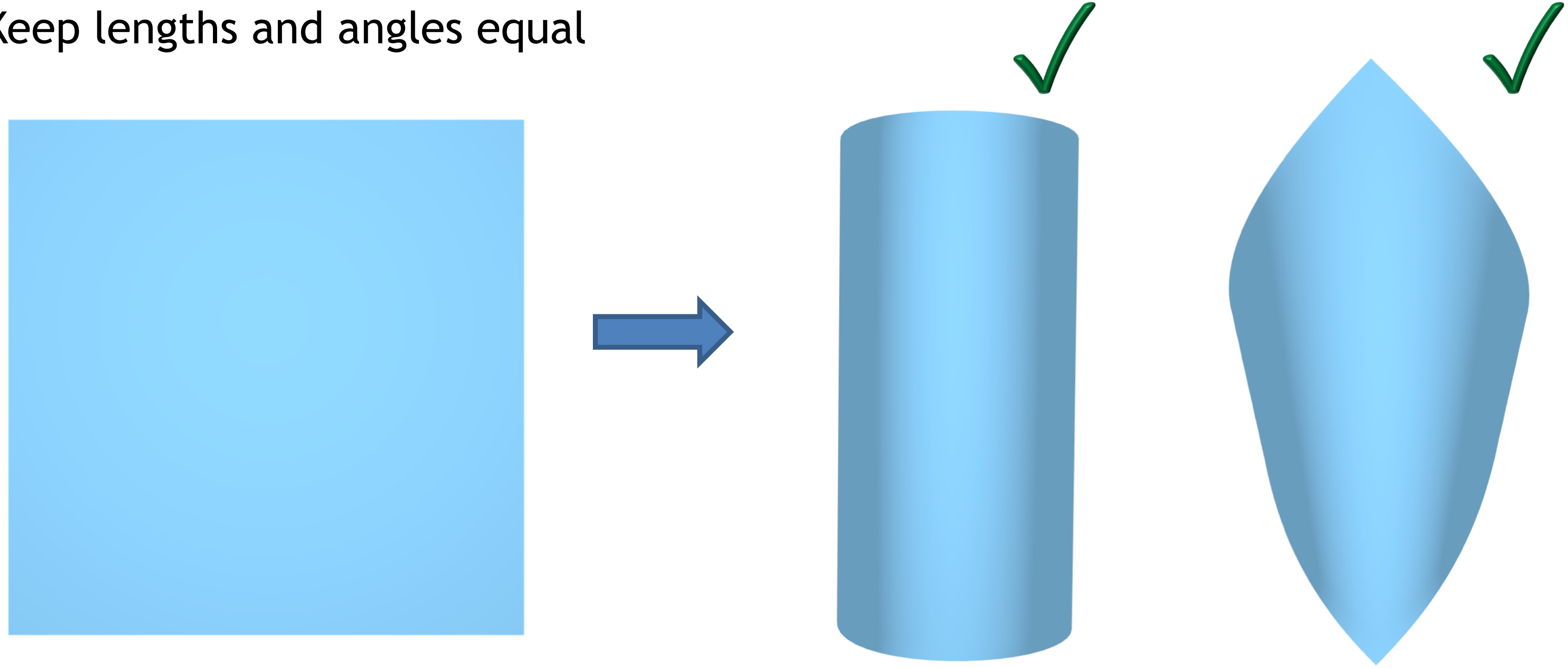


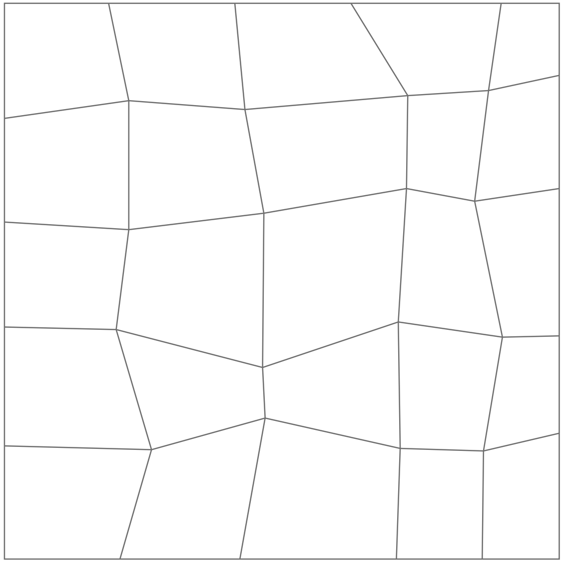
# Smooth developable

Keep lengths and angles equal



# Discretization: Locking, meshing, and constraints

Keep lengths and angles equal



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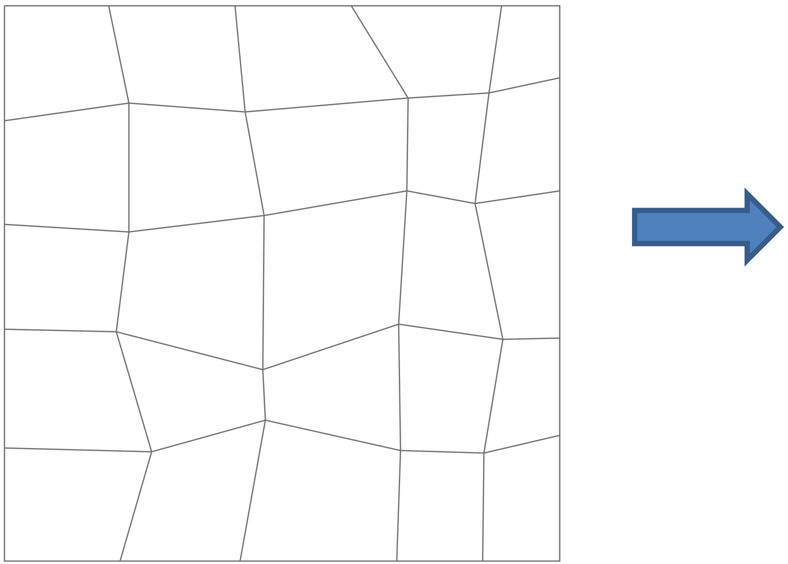
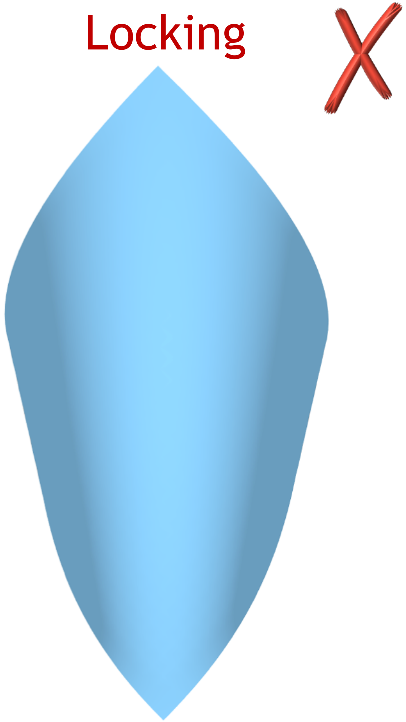


Locking

Locking



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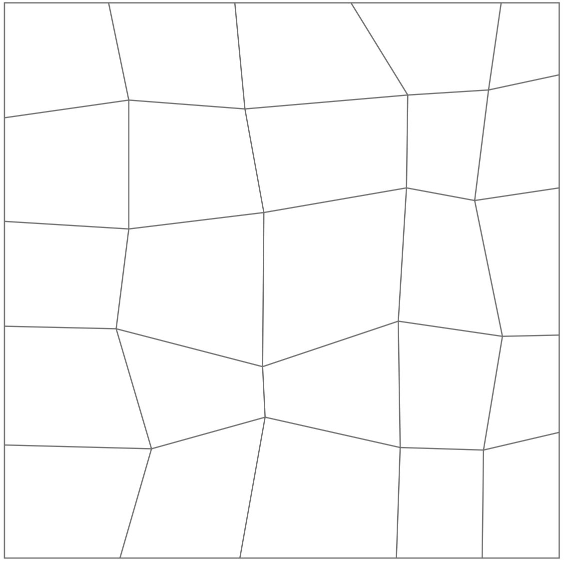
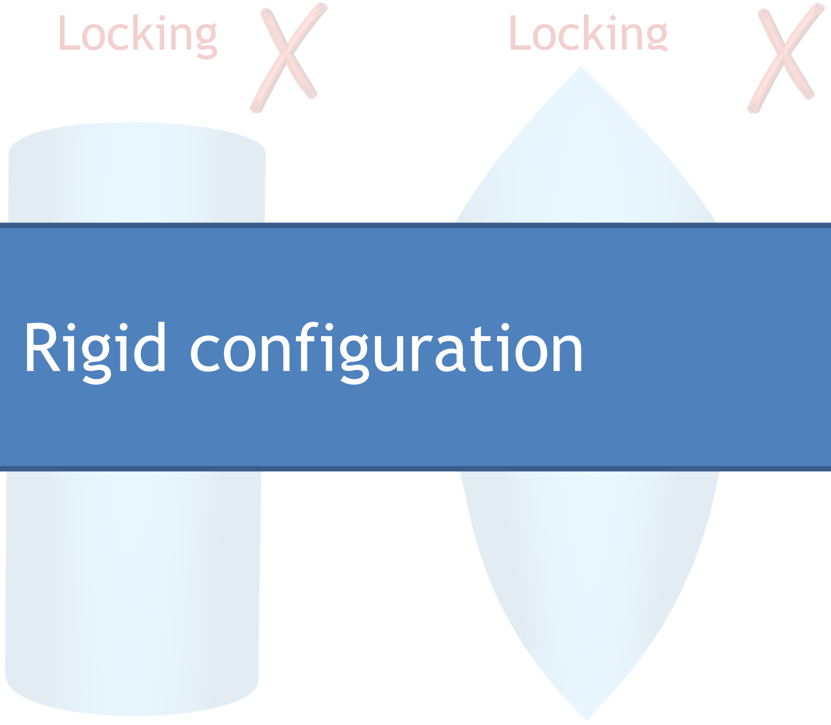


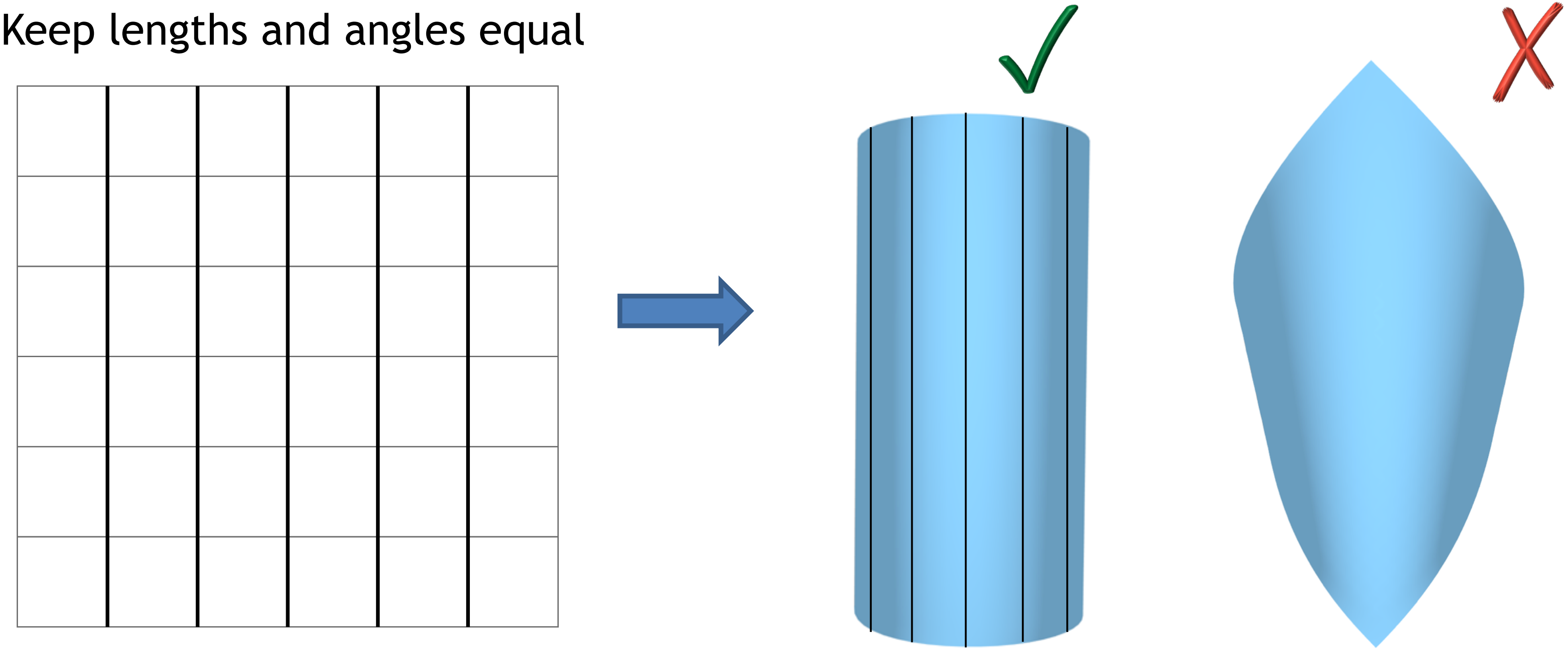



Rigid configuration

Locking: A failure of a discrete model to represent the full range of smooth deformations

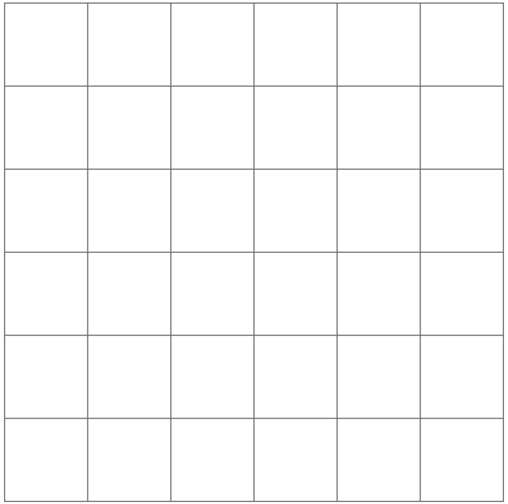
# Discretization: Locking, meshing, and constraints

Keep lengths and angles equal



# Soft penalty?

- Use soft penalty with a parameter $\rho > 0$
  - How to set the penalty?
  - Small penalty, not precise
  - Large penalty suffers from:
    - Locking
    - Slow optimization



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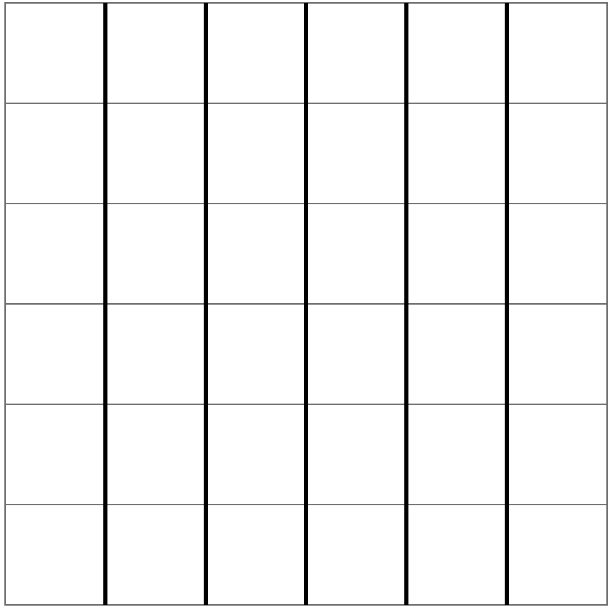
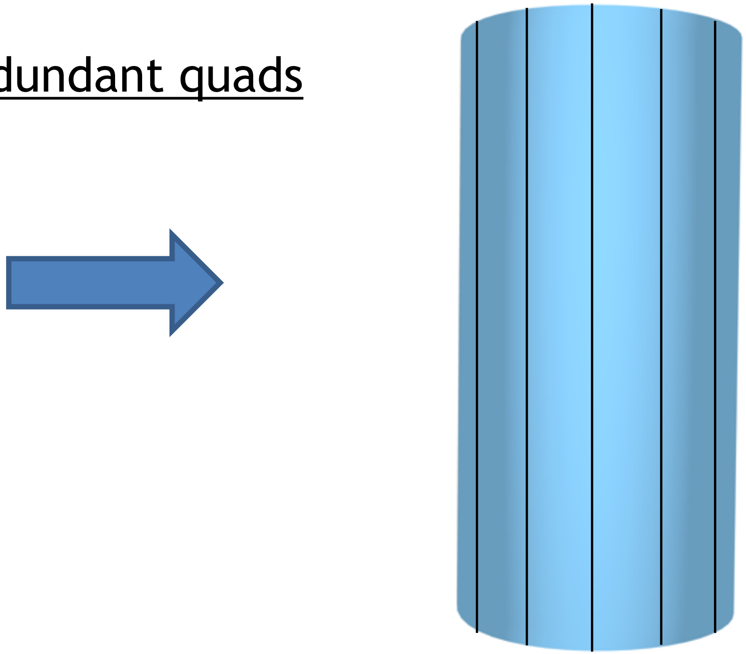
Keep lengths and angles equal



Redundant quads



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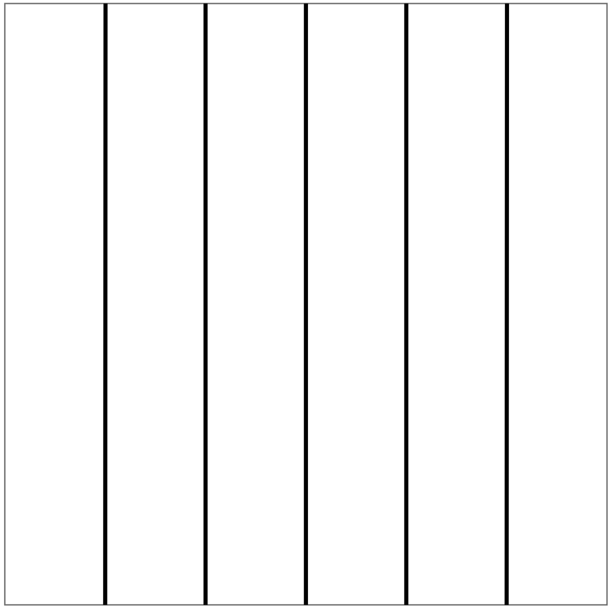
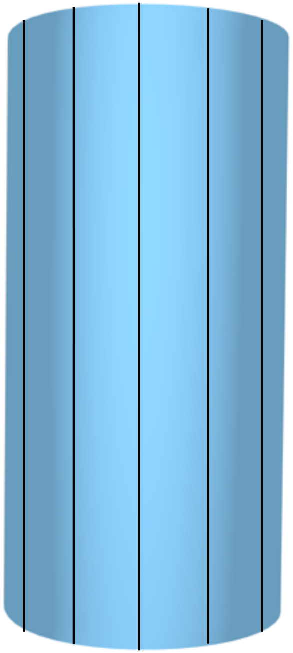
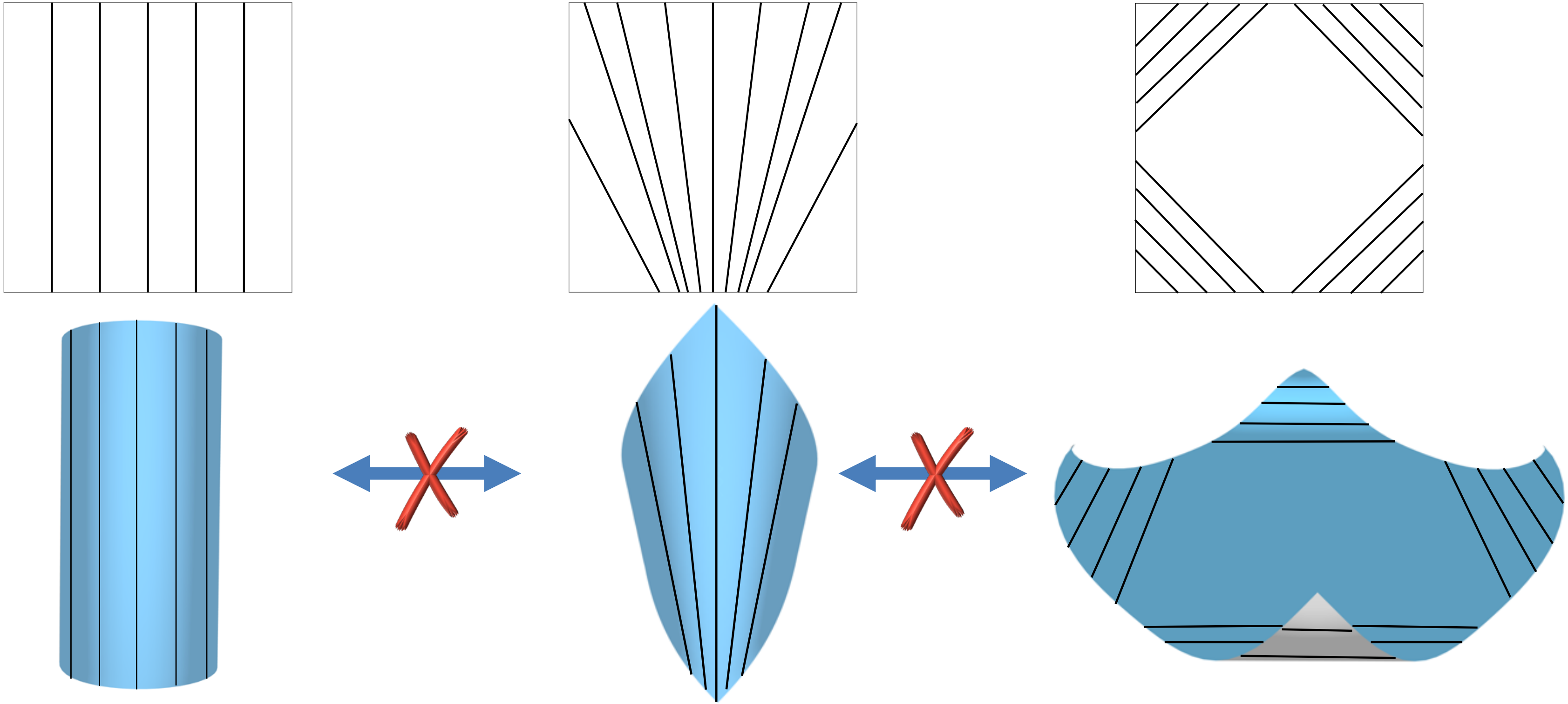


Planar quad strip

[Sauer 1970]
[Pottmann and Wallner 2001]
[Liu et al. 2007]

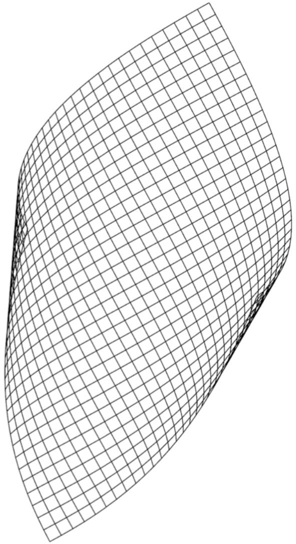


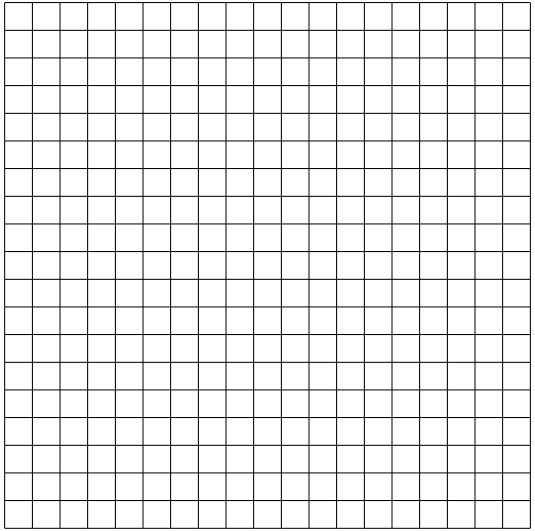
# Planar strips -exact isometry but locked

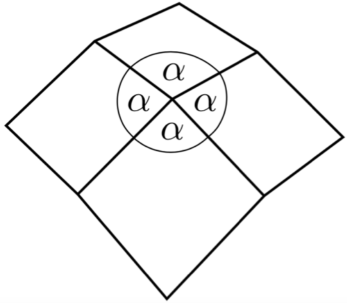


# Isometry at the limit, but doesn't lock

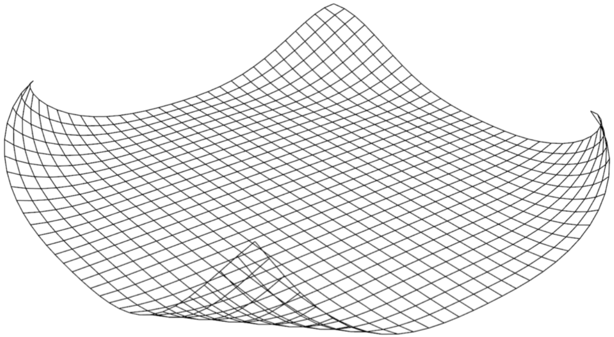
Fixed intrinsic grid meshing with a set of angle constraints








[Rabinovich et al. 2018]



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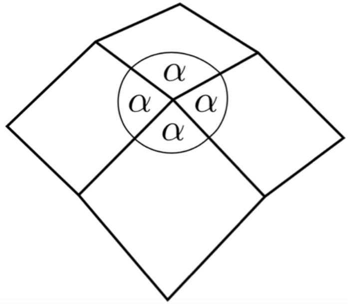
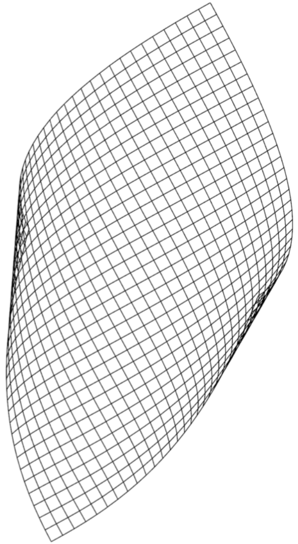
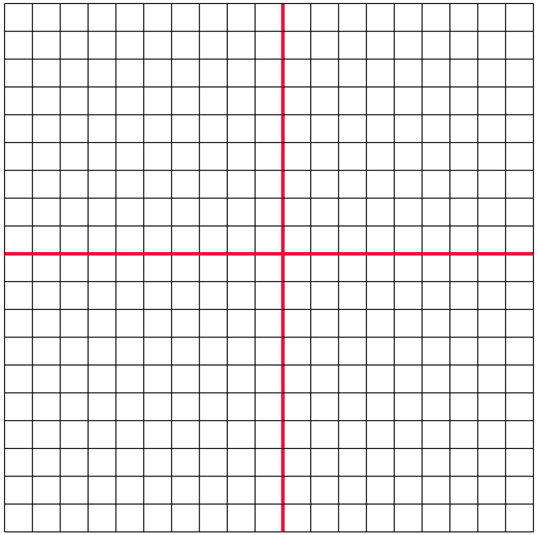
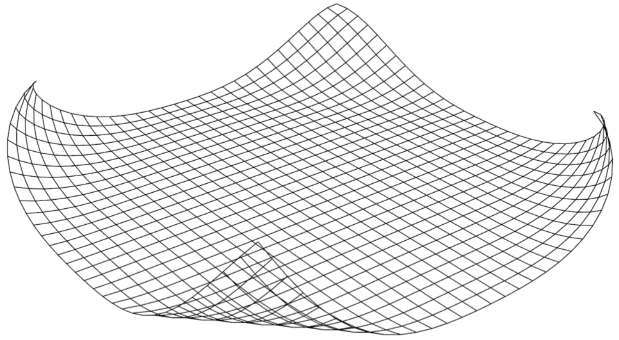



[Rabinovich et. al 2018]

Fixed edge lengths



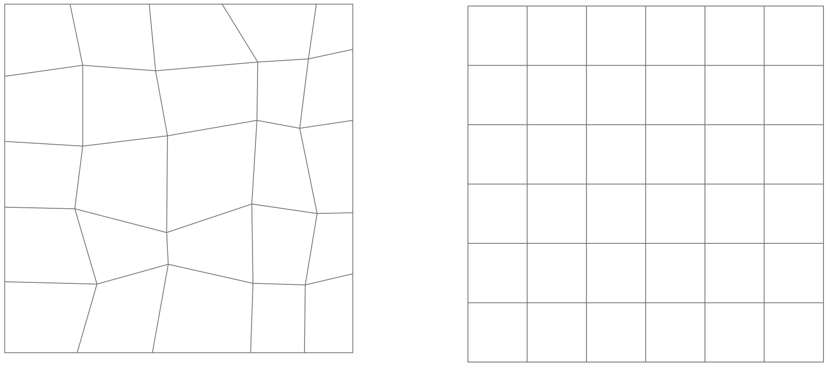




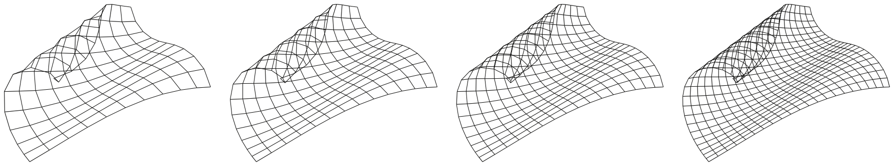
# Discrete model – lessons learned

- No free lunch
  - A discretization cannot maintain all smooth properties

- Meshing matters



# Discrete model - desiderata



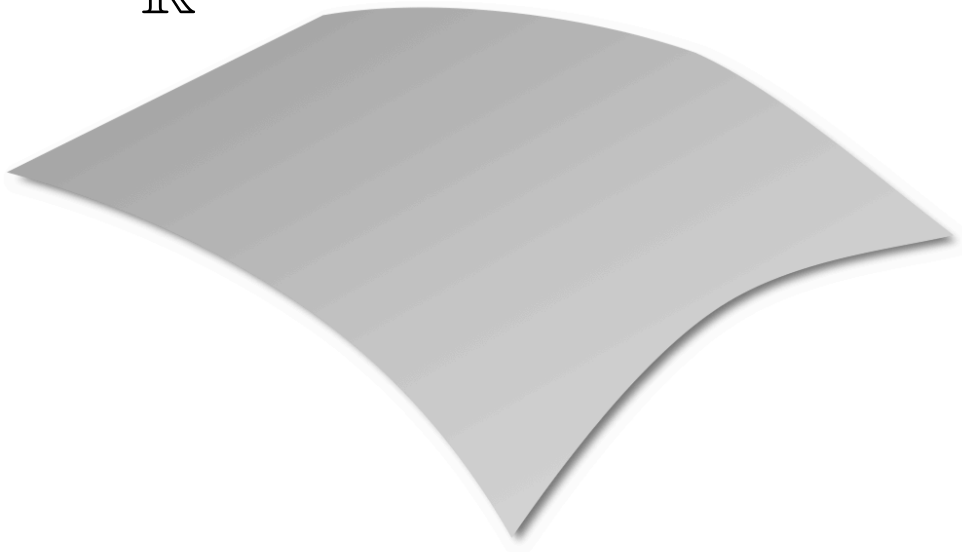
- Precise
  - Convergence at the limit
  - Similar **discrete** structure and rigidity
- Fast
  - Well defined objective and constraints we understand

"It is far, far easier to make a correct program fast than it is to make a fast program correct." - Herb Sutter

# Differential Geometry: Primer

# Surfaces, Parametric Form

$\mathbb{R}^3$



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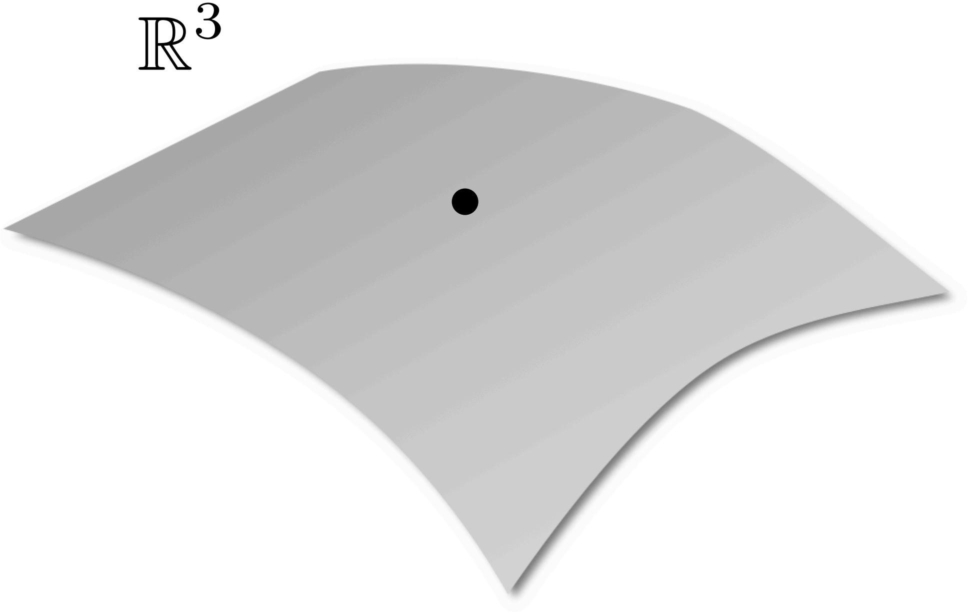

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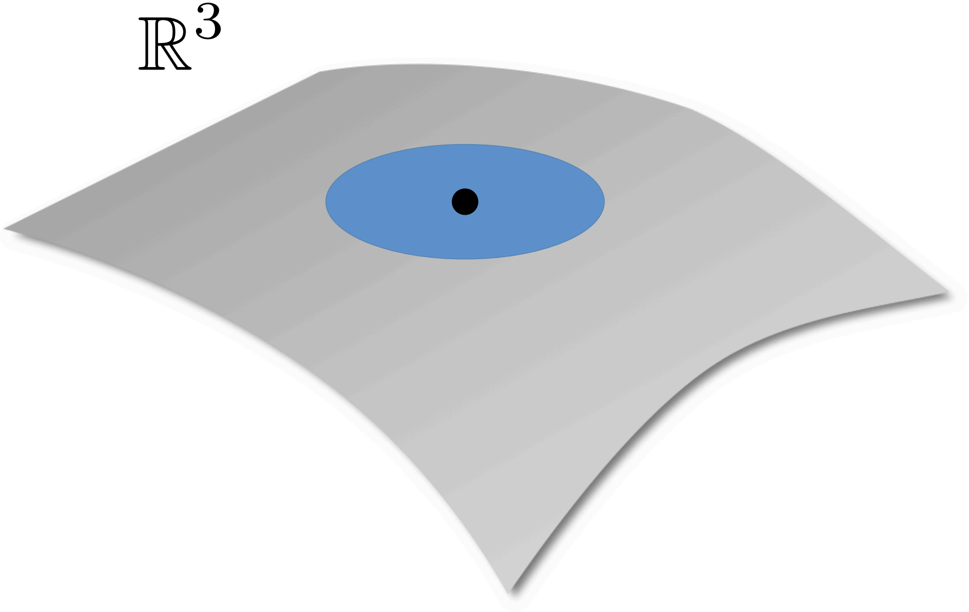

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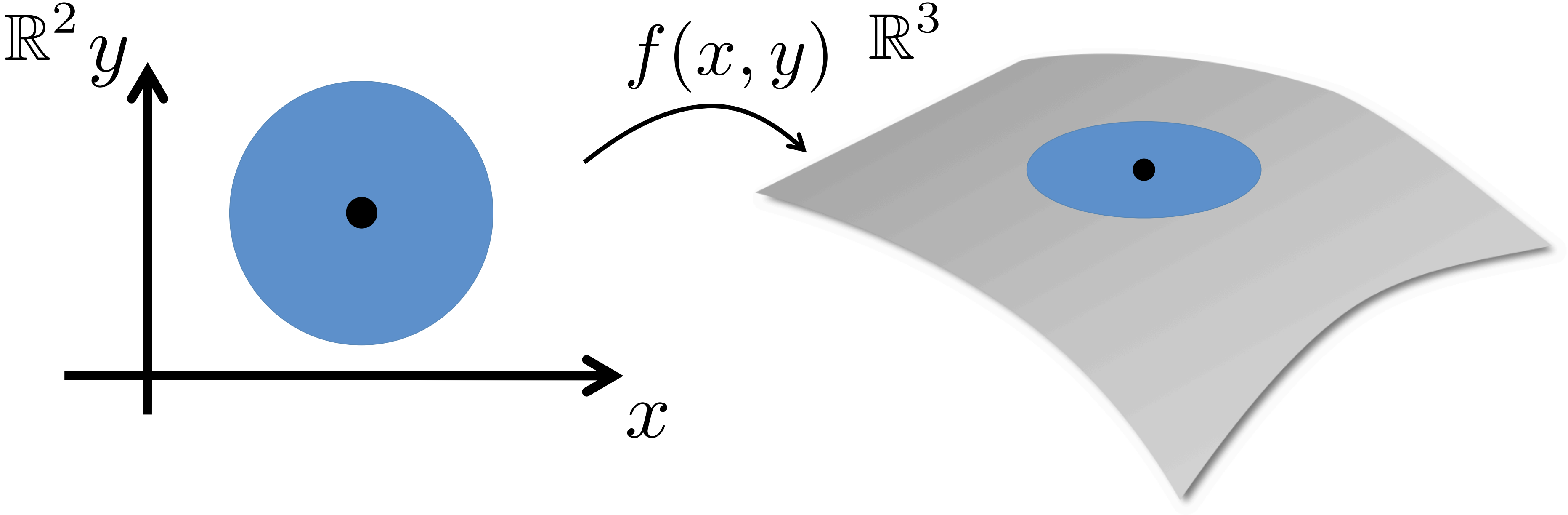

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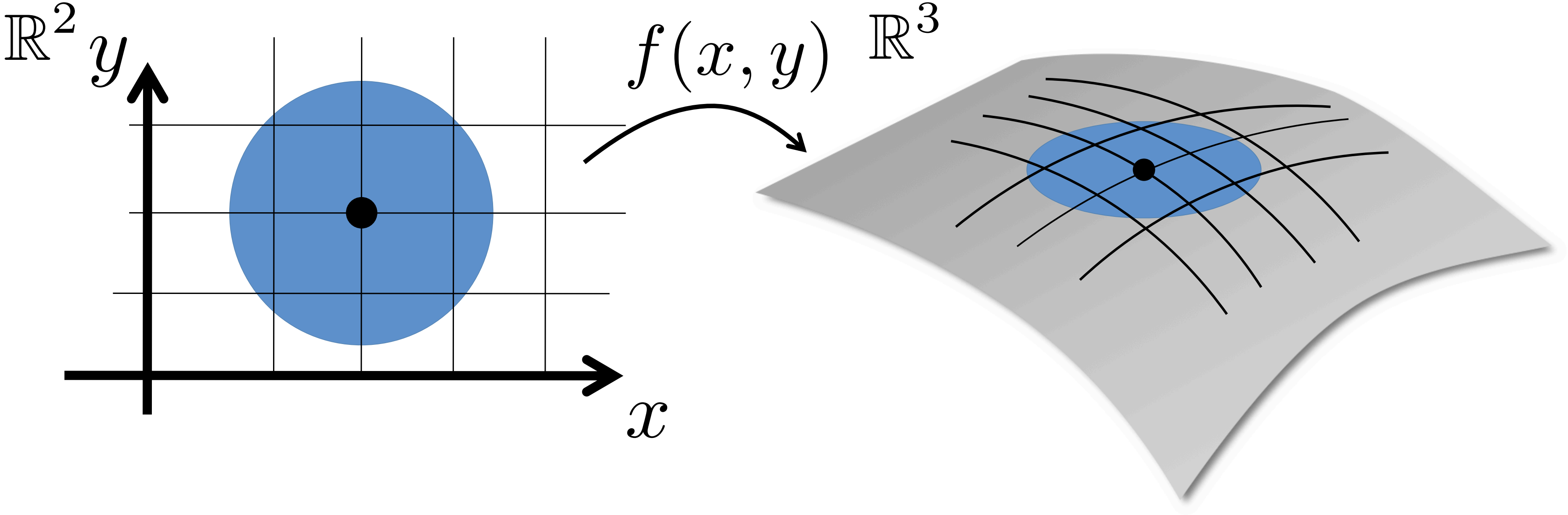

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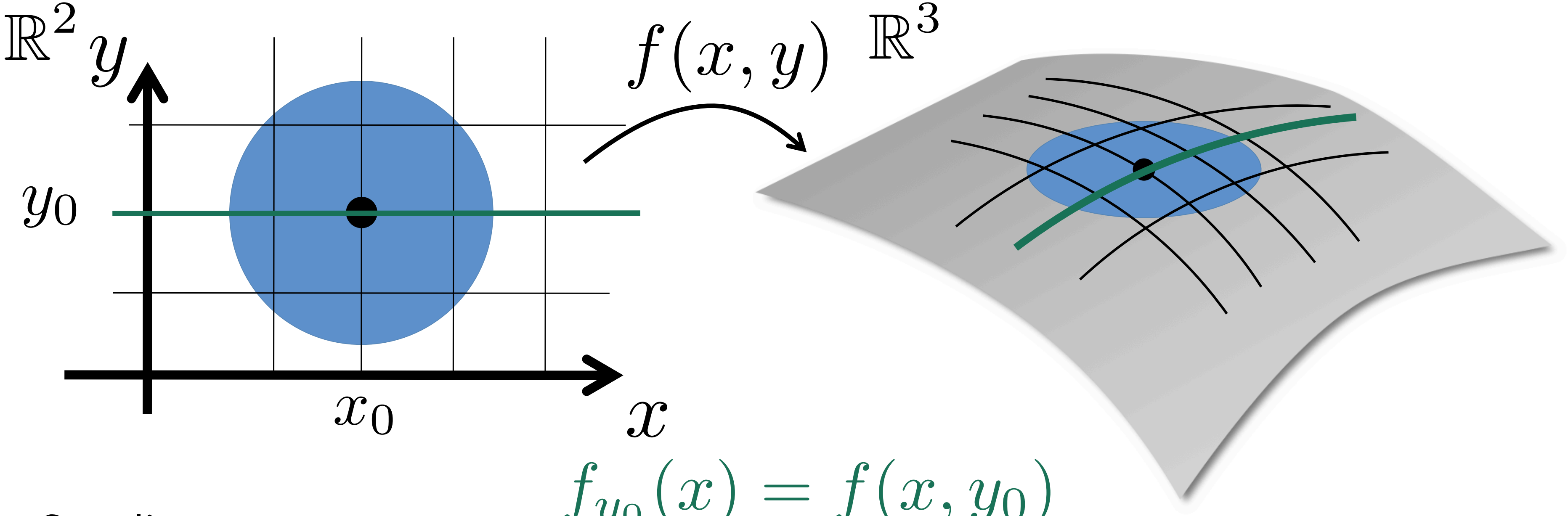
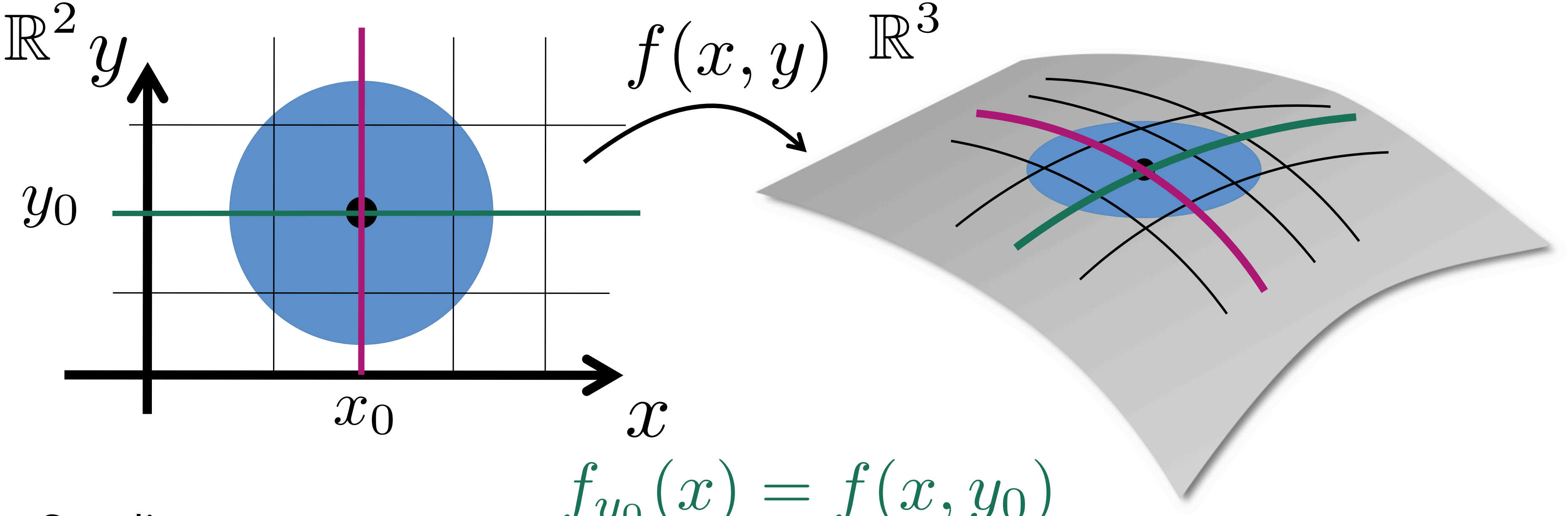



Coordinate curves:

$$f_{y_0}(x) = f(x, y_0)$$

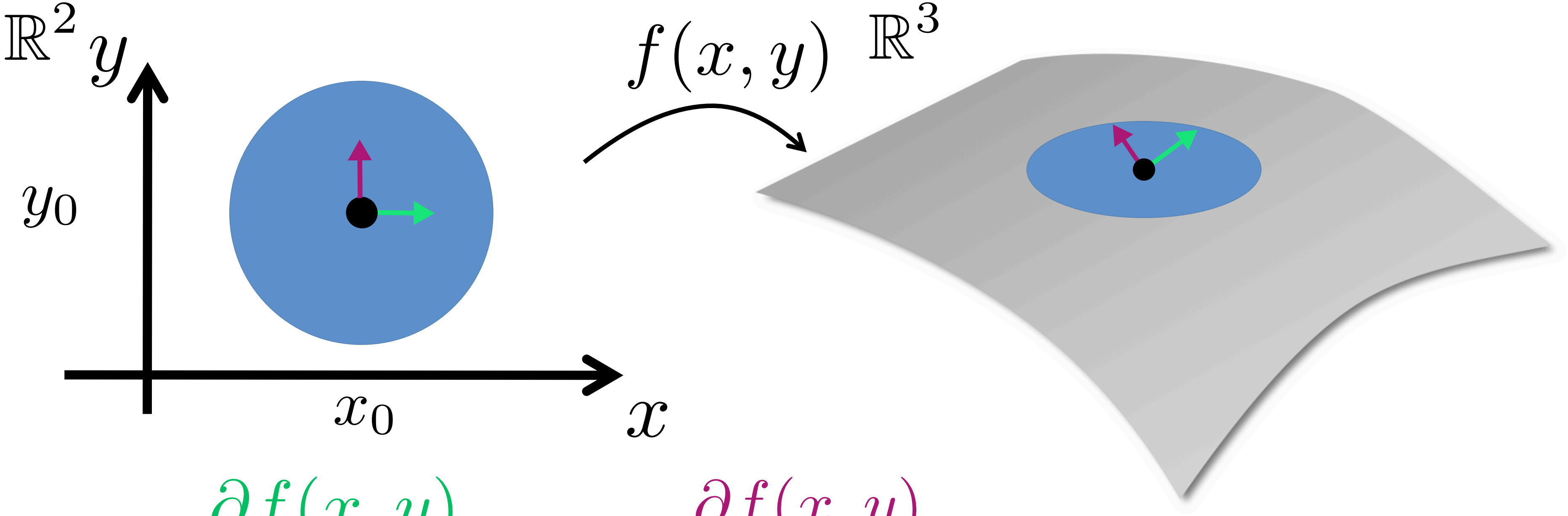
# Surfaces, Parametric Form




Coordinate curves:

$$f_{y_0}(x) = f(x, y_0)$$
$$f_{x_0}(y) = f(x_0, y)$$

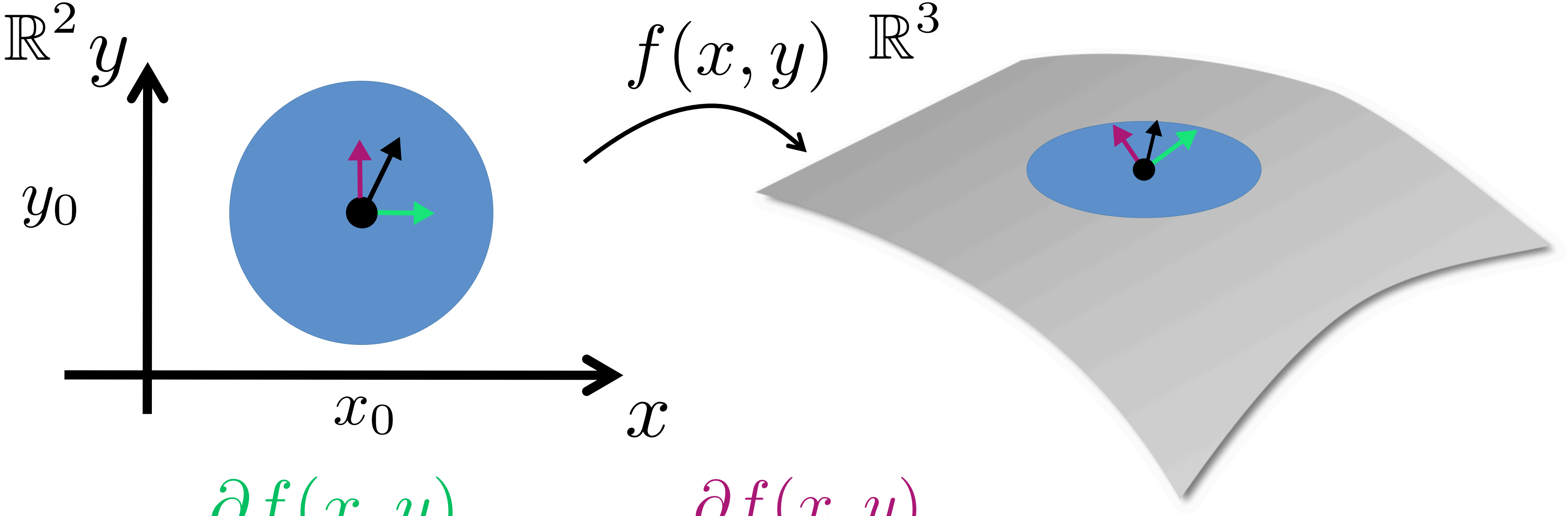
# Surfaces, Parametric Form




$$f_x = \frac{\partial f(x,y)}{\partial x} \qquad f_y = \frac{\partial f(x,y)}{\partial y}$$

Generally not orthogonal
Never parallel

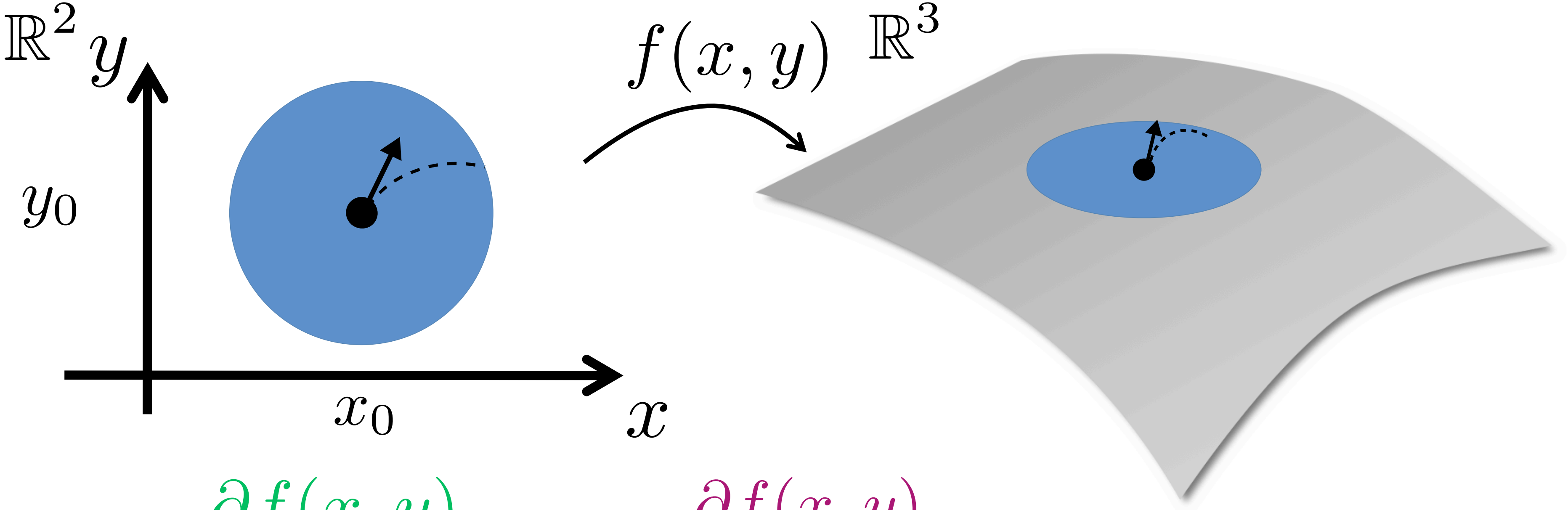
# Surfaces, Parametric Form




$$f_x = \frac{\partial f(x,y)}{\partial x} \qquad f_y = \frac{\partial f(x,y)}{\partial y}$$

Span a tangent plan
Tangent vectors as linear combination

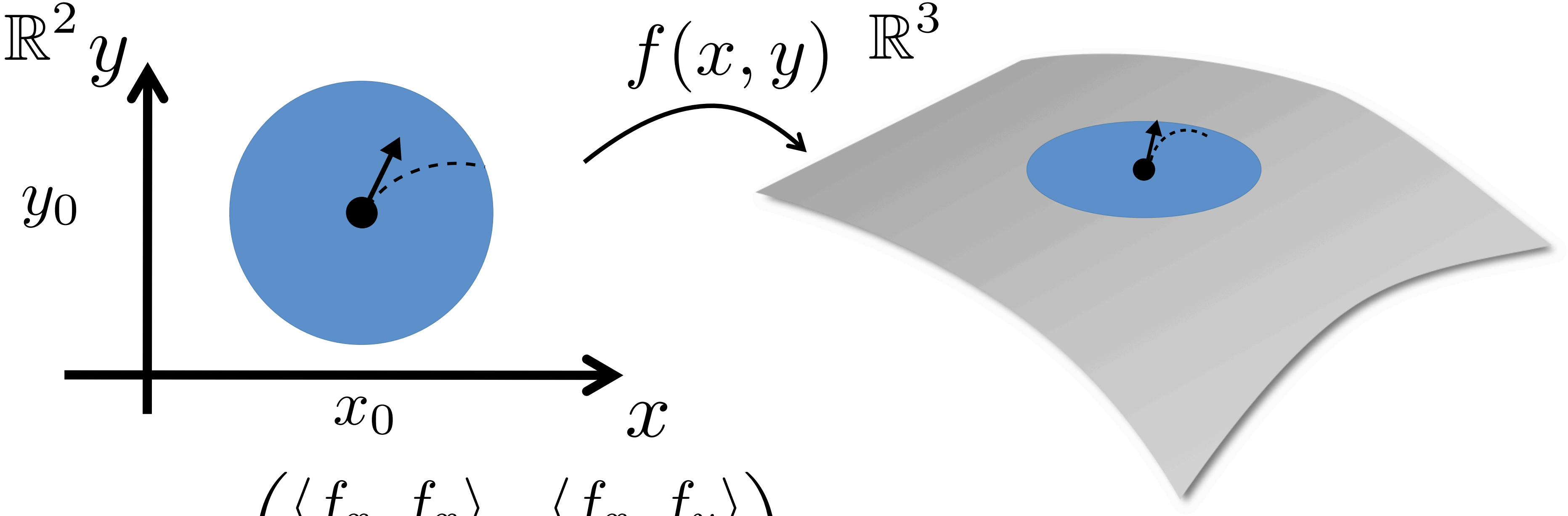
# Surfaces, Parametric Form




$$f_x = \frac{\partial f(x,y)}{\partial x} \qquad f_y = \frac{\partial f(x,y)}{\partial y}$$

Generate a curve by a smooth choice of tangents

# Surfaces, Parametric Form



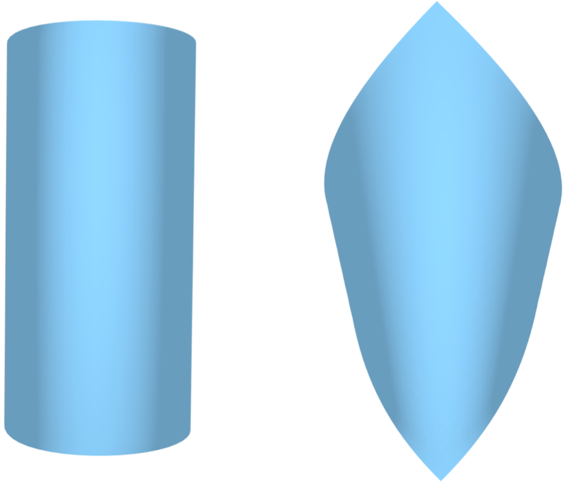

$$\mathbf{I} = \begin{pmatrix} \langle f_x, f_x \rangle & \langle f_x, f_y \rangle \\ \langle f_x, f_y \rangle & \langle f_y, f_y \rangle \end{pmatrix}$$

Measure lengths and angles

# Intrinsic and extrinsic properties

- Intrinsic properties
  - Distance and angles
- Extrinsic properties
  - Normals
  - Certain curvatures

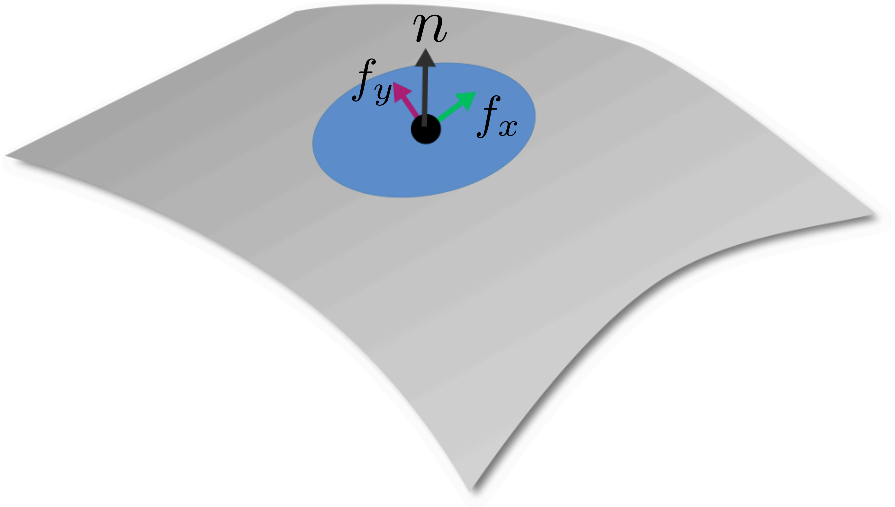
Isometric = Intrinsically the same

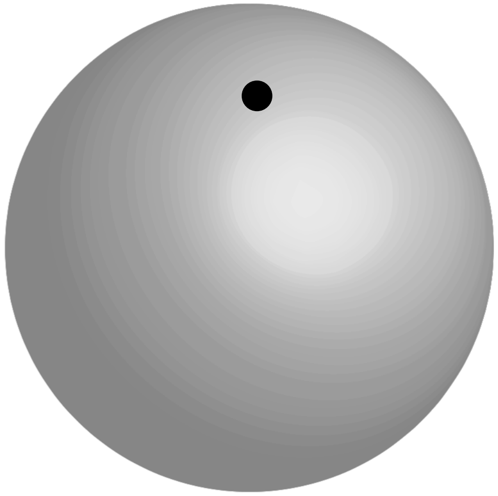


# Normals and the Gauss map

$f(x, y)$

$$n(x, y) = \frac{f_x \times f_y}{\|f_x \times f_y\|}$$

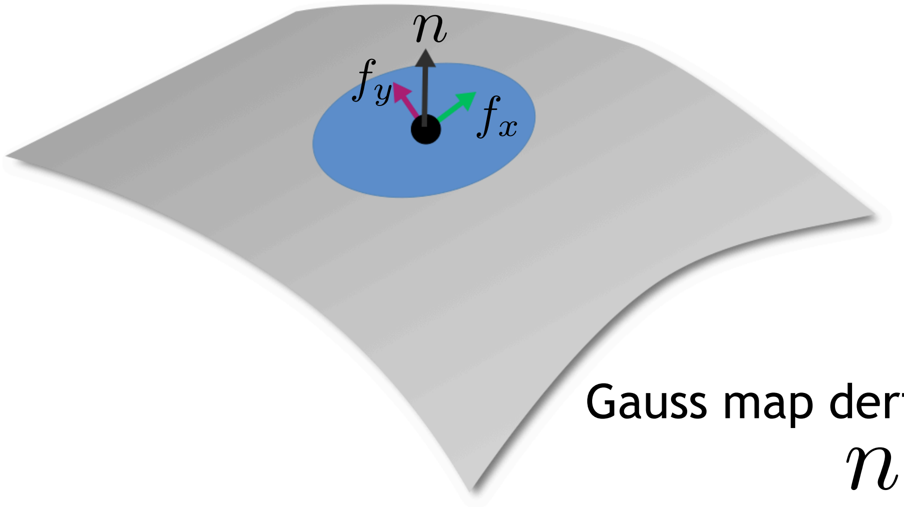


# Normals and the Gauss map

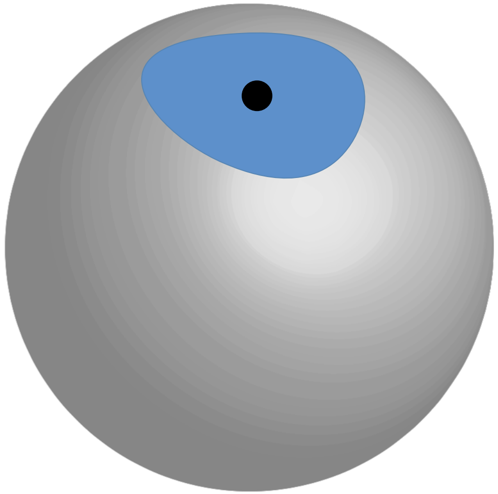
$f(x, y)$

$$n(x, y) = \frac{f_x \times f_y}{\|f_x \times f_y\|}$$




Gauss map derivatives:

$n_x, n_y$



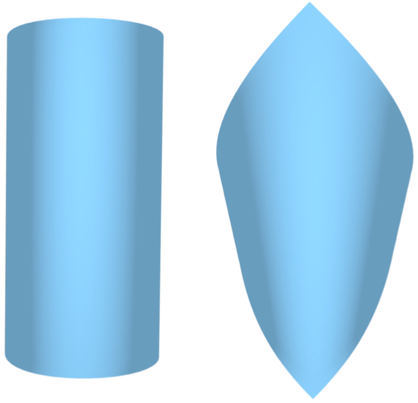
# Fundamental Forms

- First fundamental form

$$\mathbf{I} = \begin{pmatrix} \langle f_x, f_x \rangle & \langle f_x, f_y \rangle \\ \langle f_x, f_y \rangle & \langle f_y, f_y \rangle \end{pmatrix}$$

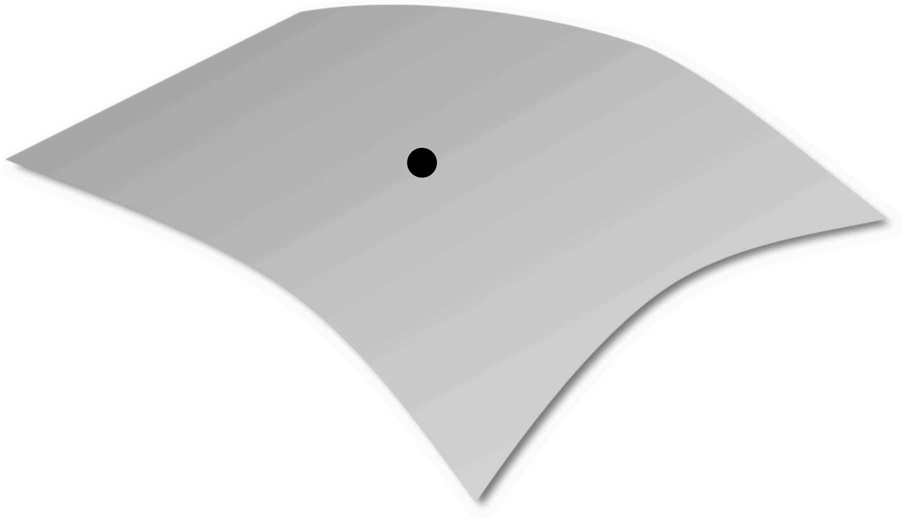
- Second fundamental form

$$\mathbf{II} = \begin{pmatrix} \langle -f_x, n_x \rangle & \langle -f_y, n_x \rangle \\ \langle -f_y, n_x \rangle & \langle -f_y, n_y \rangle \end{pmatrix}$$

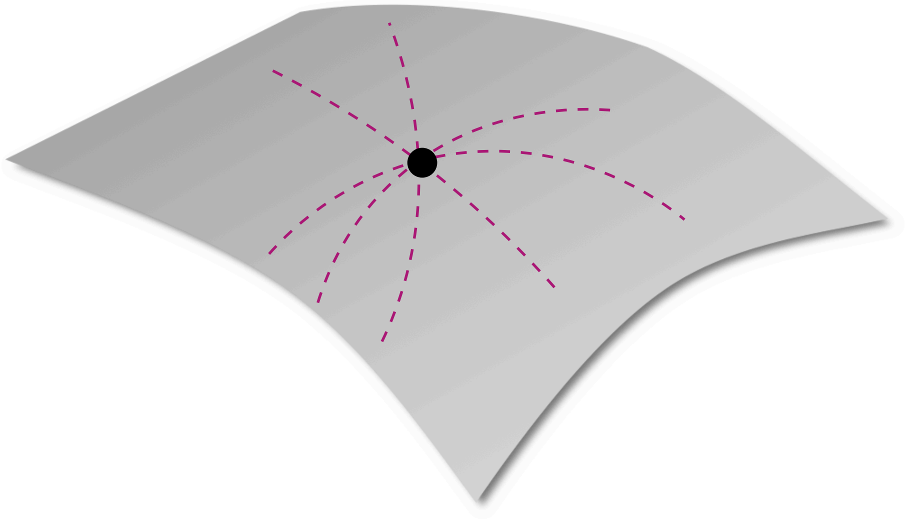


- Together define the surface up to rigid motion (if they are compatible)

# Curvature

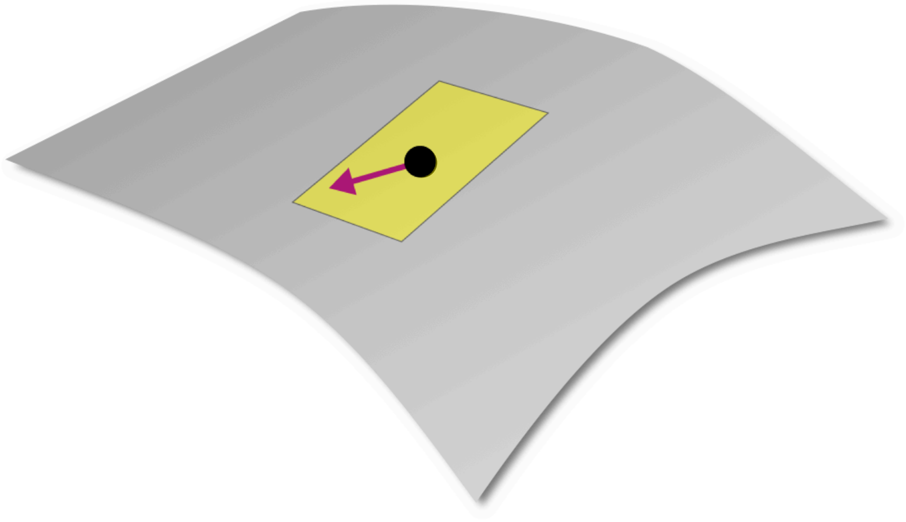


# Curvature



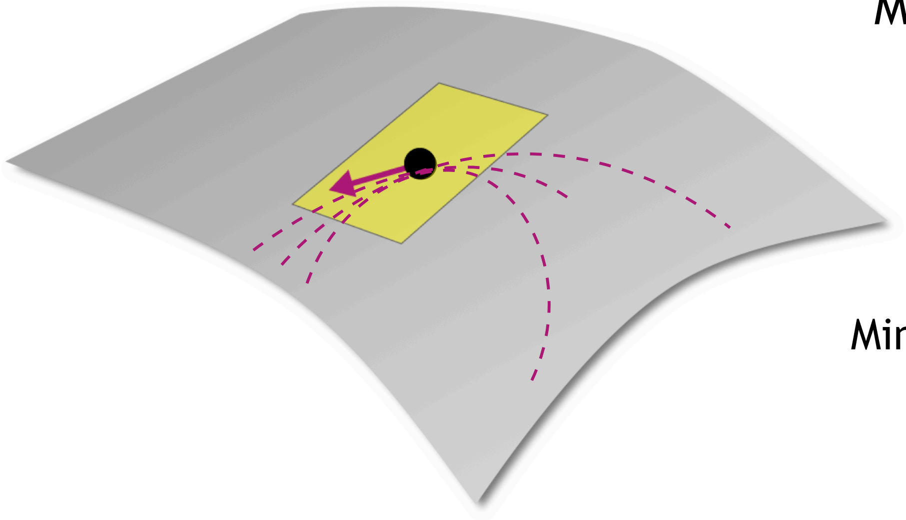
Many curves through a point

# Curvature



Fix a direction (tangent)

# Normal Curvature



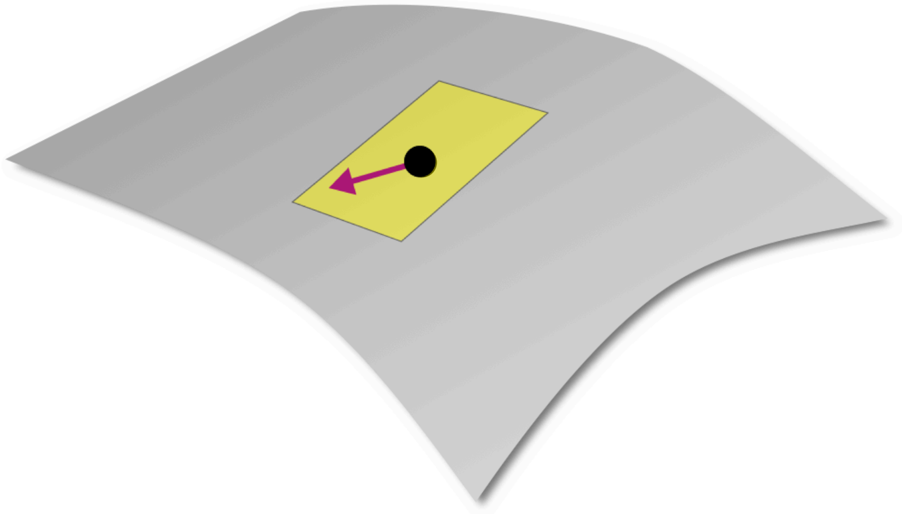
Fix a direction (tangent)

Many surface curves with the same tangent

<u>Normal Curvature</u>

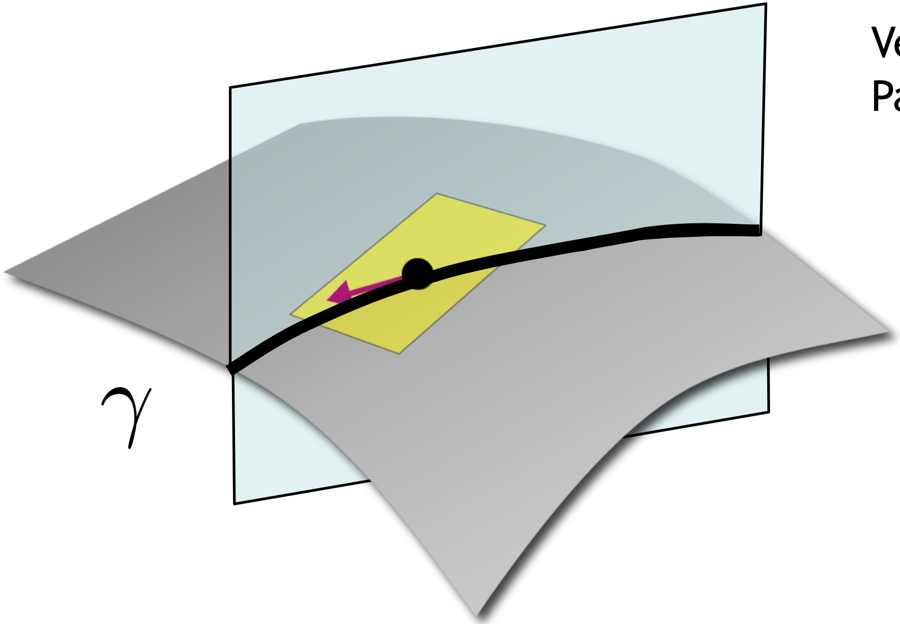
Minimum curvature of curves in a given direction

# Computing the Normal Curvature



Vector $\mathbf{t}$ in the tangent plane

# Computing the Normal Curvature

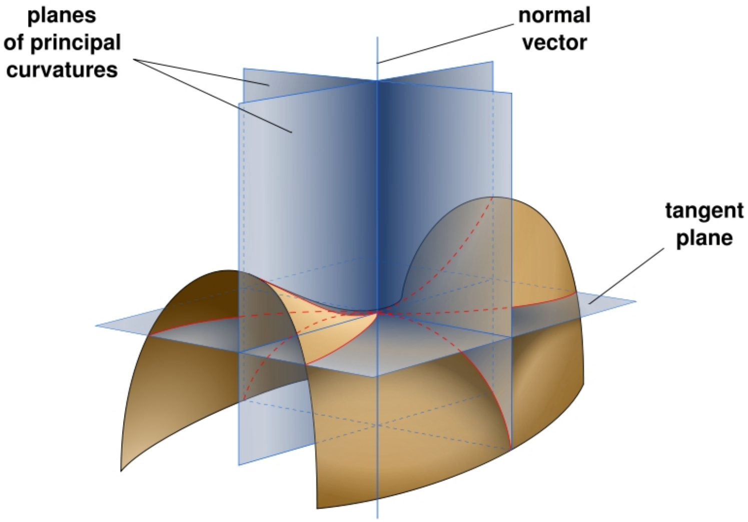



Vector $\mathbf{t}$ in the tangent plane

Pass a plane spanned by $\mathbf{t}$ and the normal $\mathbf{n}$

$$k_n(t) = k(\gamma(\mathbf{p}))$$

# Principal Directions



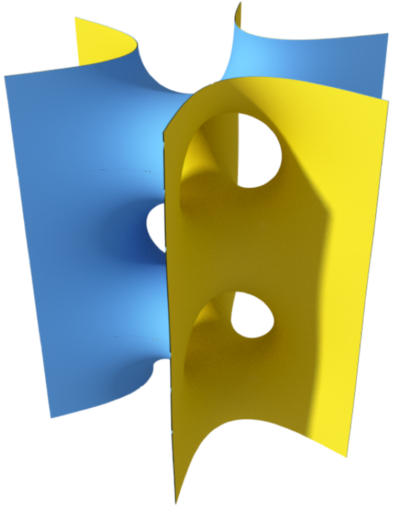

**Euler's Theorem:** Planes of maximal and minimal normal curvature are orthogonal

# Surface Curvatures

- Principal curvatures
  - Minimal curvature $\kappa_1 = \kappa_{\min} = \min_{\varphi} \kappa_n(\varphi)$
  - Maximal curvature $\kappa_2 = \kappa_{\max} = \max_{\varphi} \kappa_n(\varphi)$
- Mean curvature $H = \frac{\kappa_1 + \kappa_2}{2}$
- Gaussian curvature $K = \kappa_1 \cdot \kappa_2$

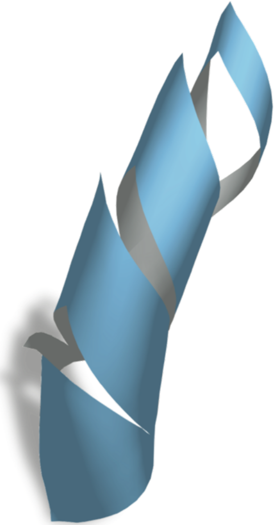
# Classification of surfaces by curvatures

$$H = 0$$



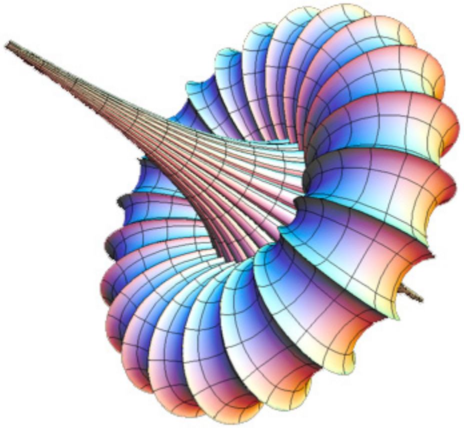
Minimal surfaces

$$K = 0$$



Developable

$$K = -1$$

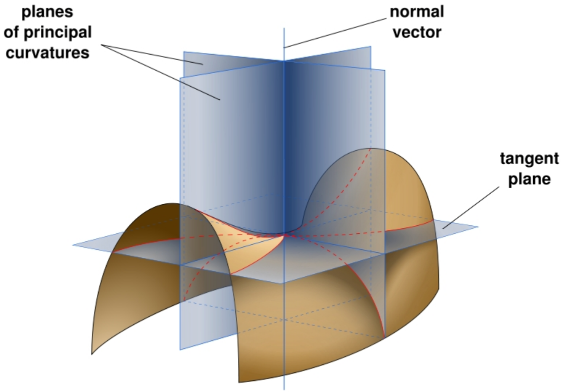


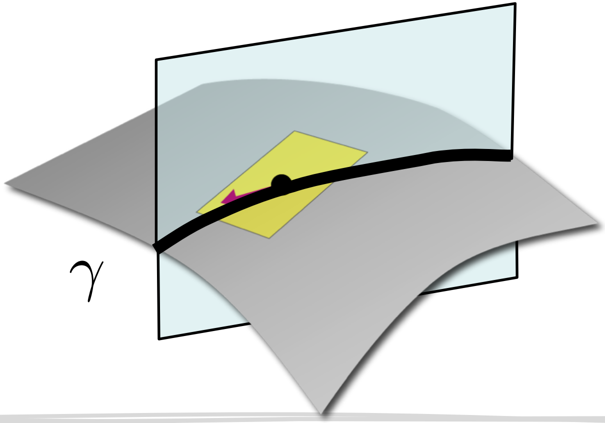
Pseudospherical

Virtual Math Museum, http://virtualmathmuseum.org/index.html

# Classification of curves by curvature

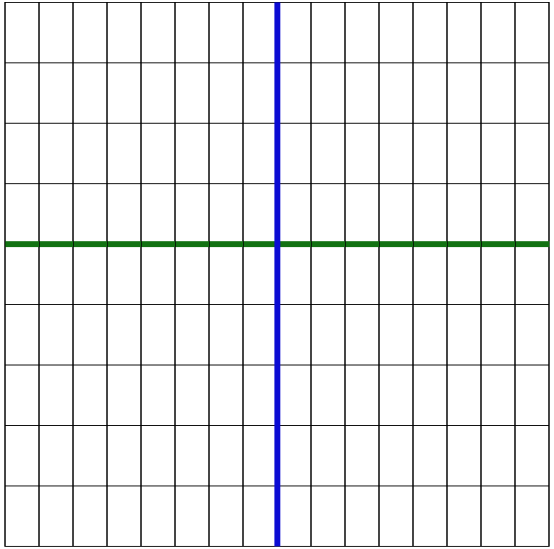
- Curvature line: Aligned to a principle curvature
- Geodesic: Curvature equals normal curvature
- Asymptotic curve: 0 normal curvature



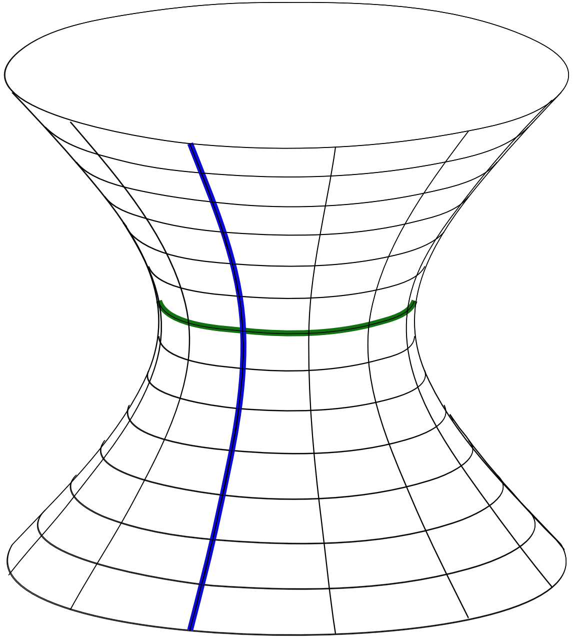



# Classification of parameterizations

$\mathbb{R}^2$ Coordinate curves as special curves $\mathbb{R}^3$

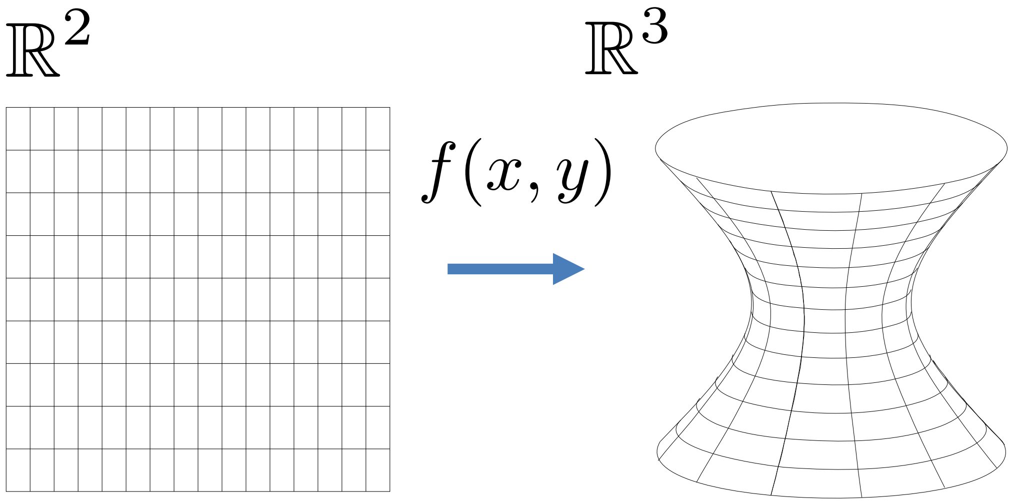


$$f(x, y)$$



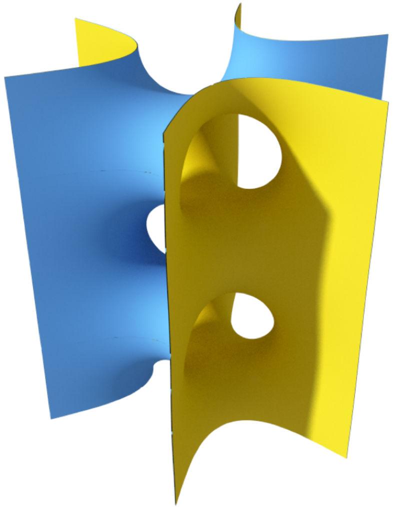
# Classification of parameterizations by coordinate curves

- Curvature lines
- Geodesics
- Asymptotic lines
- Chebyshev nets



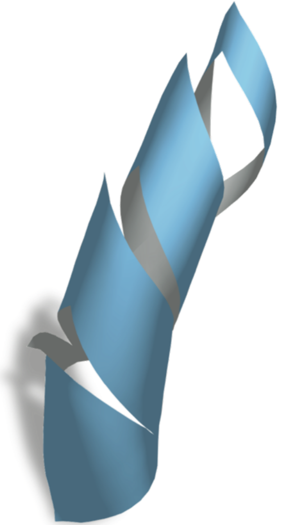
# Classification of surfaces by curvatures

$$H = 0$$



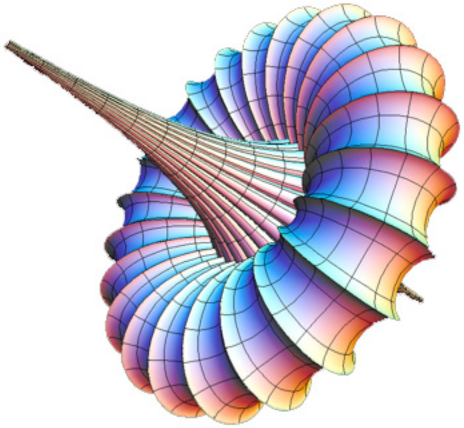
Minimal surfaces

$$K = 0$$



Developable

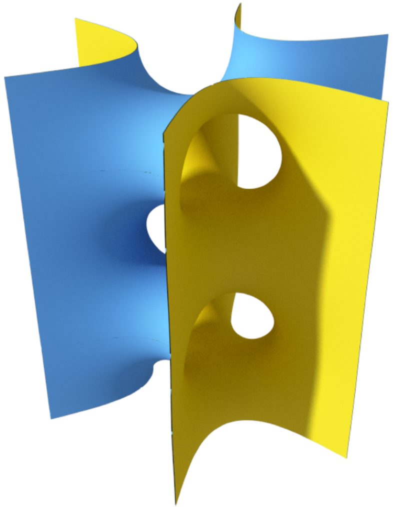
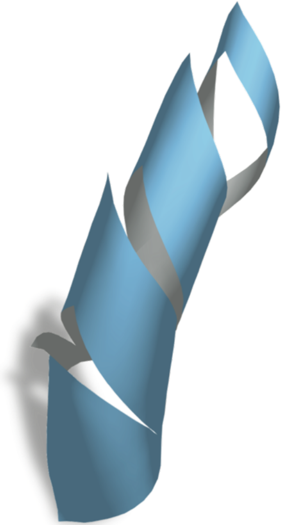
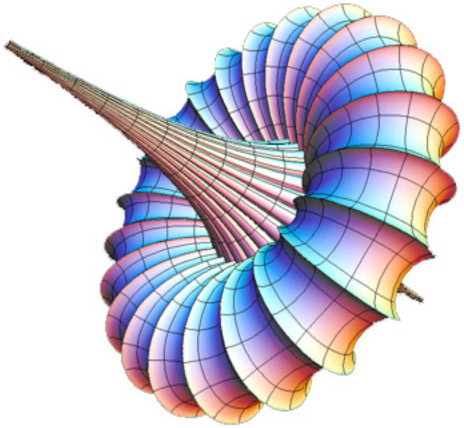
$$K = -1$$



Pseudospherical

Virtual Math Museum, http://virtualmathmuseum.org/index.html

# Classification of surfaces by parameterizations

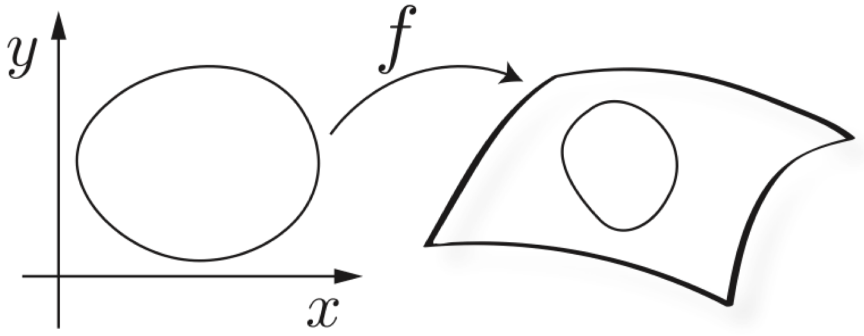
| Orthogonal Asymptotic | Orthogonal Geodesics | Asymptotic Chebyshev |
| --- | --- | --- |
|  |  |  |
| Minimal surfaces | Developable | Pseudospherical |

Virtual Math Museum, http://virtualmathmuseum.org/index.html

# Discrete Nets

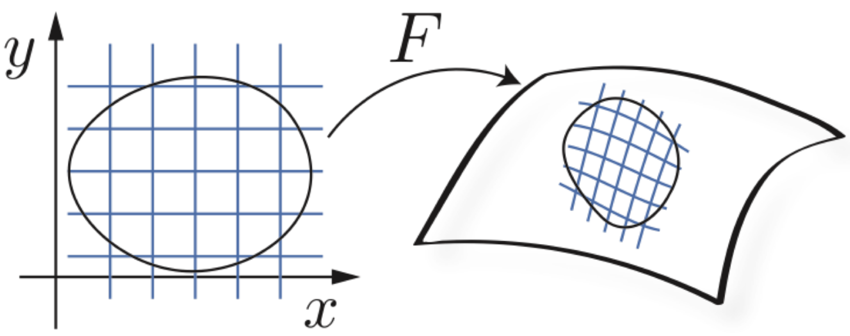
# Nets in discrete differential geometry

Parameterization/Smooth net




$$f : \mathbb{R}^2 \to \mathbb{R}^3$$

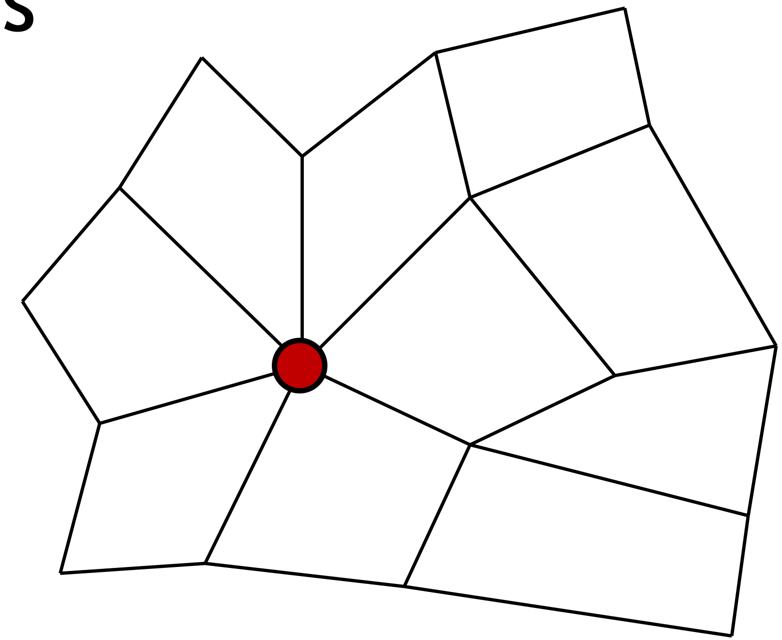
Discrete net




$$F : \mathbb{Z}^2 \to \mathbb{R}^3$$

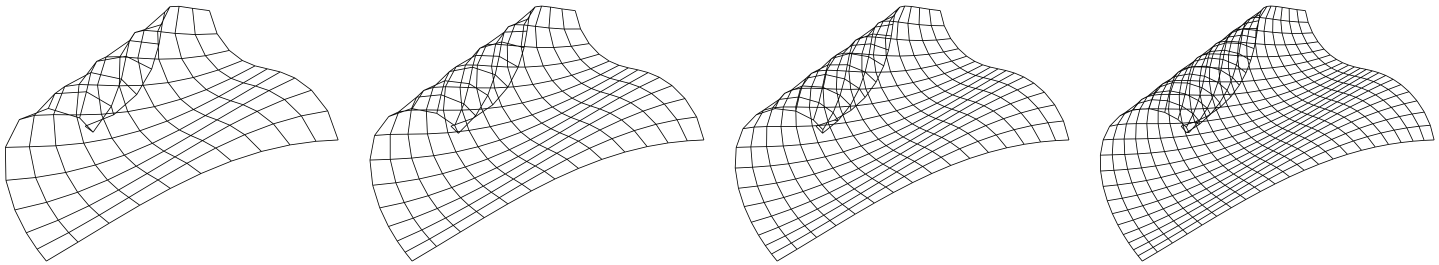
# Singular points

- General quad meshes



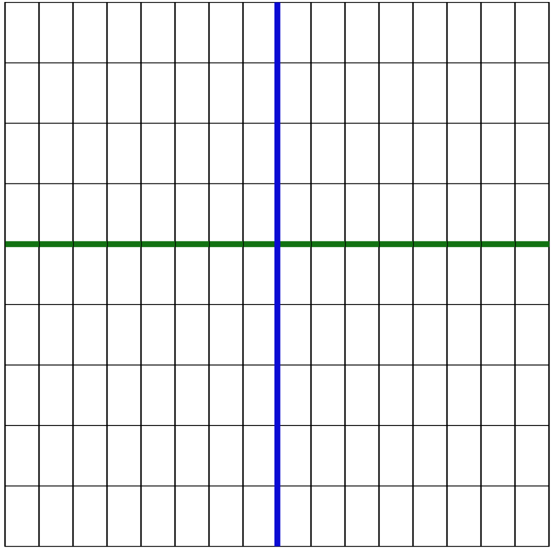
# Properties of a good discretization

- Converges to the smooth counterpart
- Simple
- Preserves structure

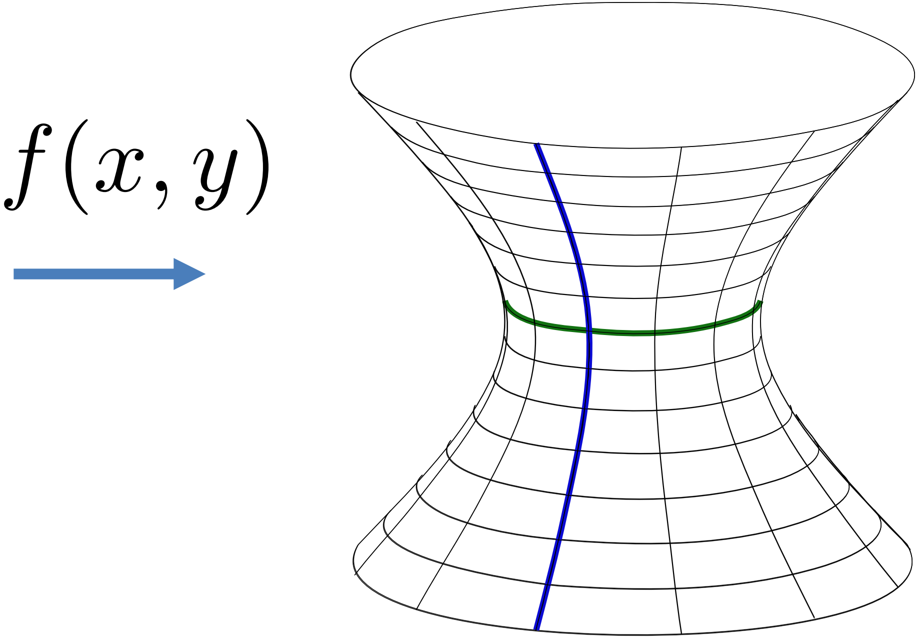


# Example – curvature line nets

$f_x, f_y$ Principle curvature directions



$f(x, y)$



# Curvature line net

<u>Smooth</u>

$$n_x \parallel f_x \, , n_y \parallel f_y$$

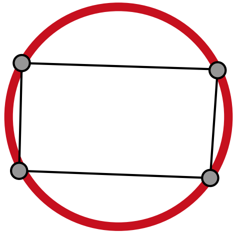
# Curvature line net

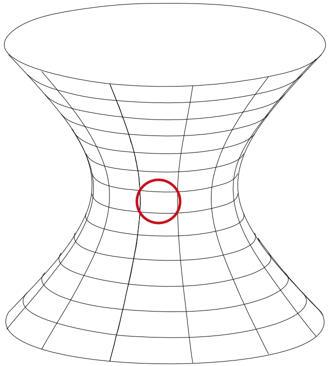
## Smooth

$$n_x \parallel f_x, n_y \parallel f_y$$

## Discrete

Quads are circular





[Martin Ralph R and Nutbourne Anthony W, 1988]
[Alexander I.Bobenko and Yuri B.Suris 2008]

# Curvature line net

## Smooth

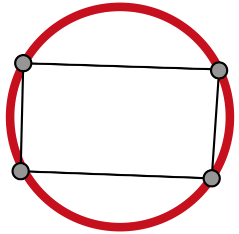
$$n_x \parallel f_x, n_y \parallel f_y$$

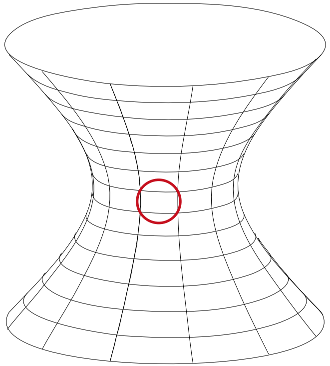
???

↔

## Discrete

Quads are circular





[Martin Ralph R and Nutbourne Anthony W, 1988]
[Alexander I.Bobenko and Yuri B.Suris 2008]

# Curvature line net

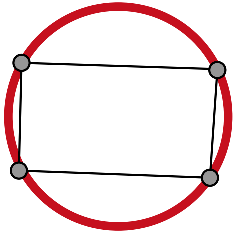
## Smooth

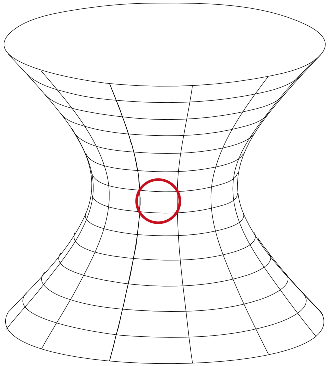
## Discrete

$$n_x \parallel f_x,\, n_y \parallel f_y$$

???

Quads are circular



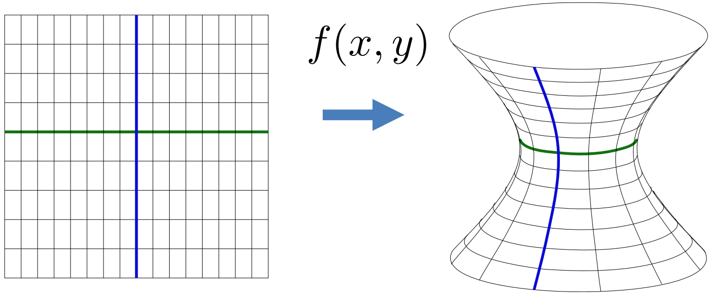


**In applications: Why not use finite differences?**

[Martin Ralph R and Nutbourne Anthony W, 1988]
[Alexander I.Bobenko and Yuri B.Suris 2008]

# Discrete curvature line mesh
# standard numerical discretization

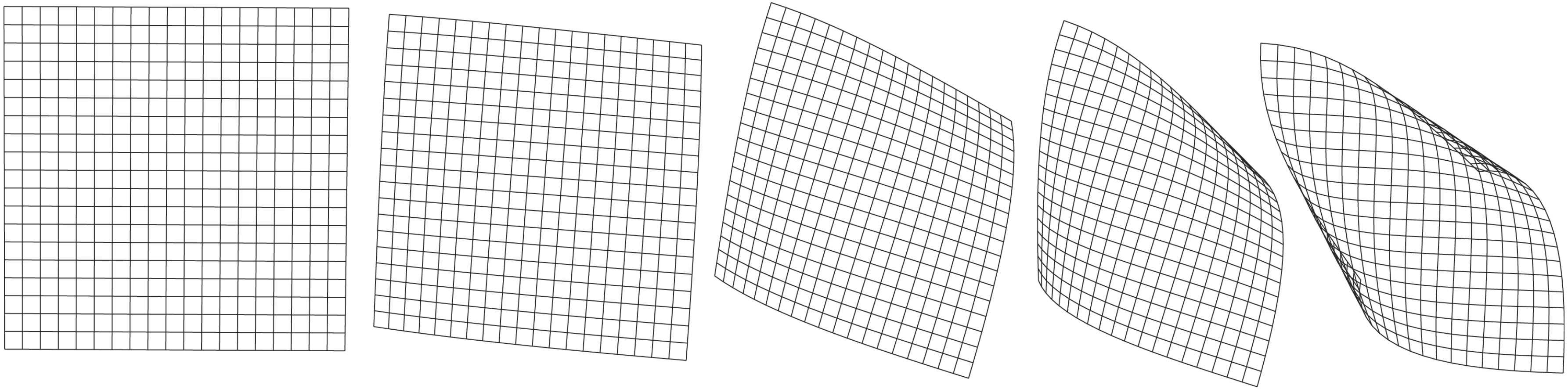
$$n_x \parallel f_x, n_y \parallel f_y$$




- Convergence? Maybe
- Simple? generally no
- How does it behave?
  - Structure/locking

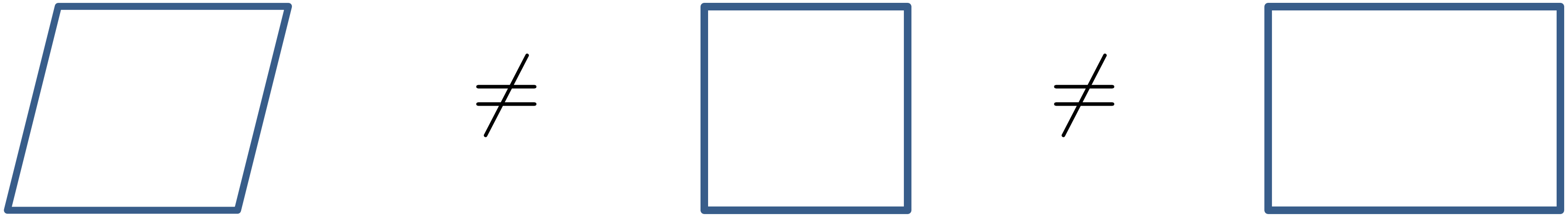
# Structure through transformations

- Study geometries using the transformations they are invariant to



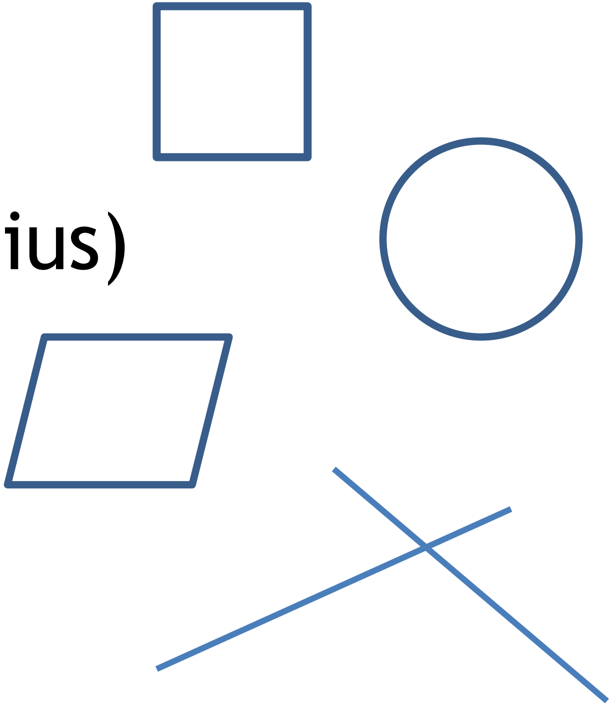
# Example - Euclidean Geometry

- Properties that are invariant under rigid motions
  - Length
  - Angles



# The Erlangen program

- Length and angles - Rigid
- Angles - Conformal (Moebius)
- Parallel lines - Affine
- Incidence - Projective

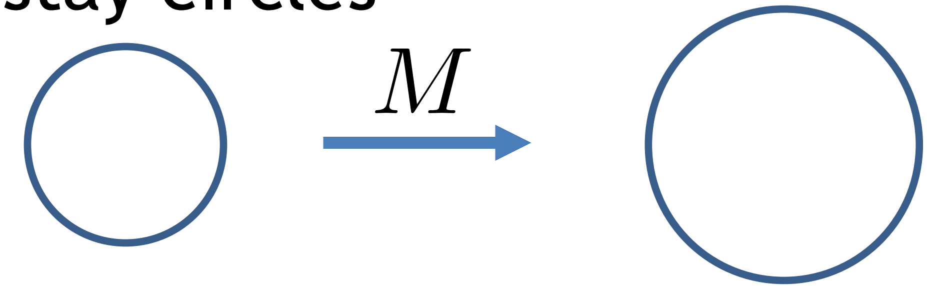


# Deformations on curvature line net

- Invariant to Moebius transformations

  $f$ curvature line net $\Longleftrightarrow$ $M(f)$ curvature line net

  - Circles stay circles

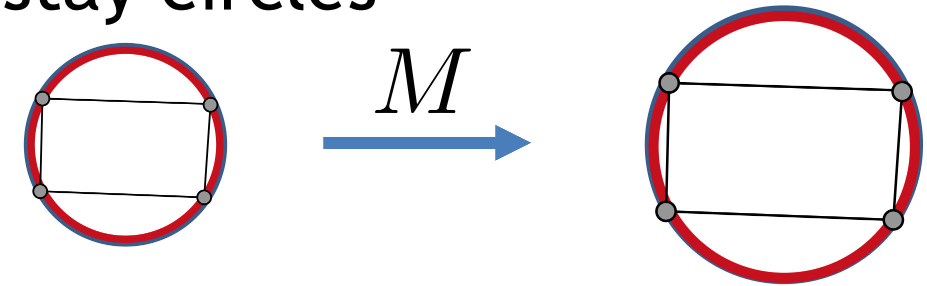


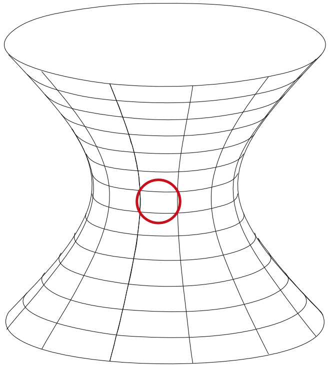
# Deformations on curvature line net

- Invariant to Moebius transformations

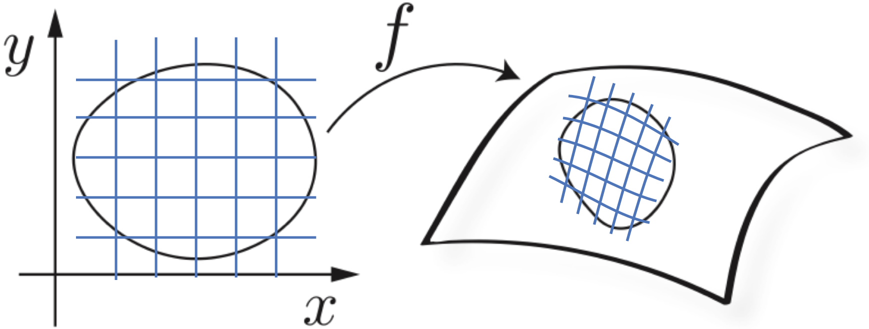
$f$ curvature line net ⟺ $M(f)$ curvature line net

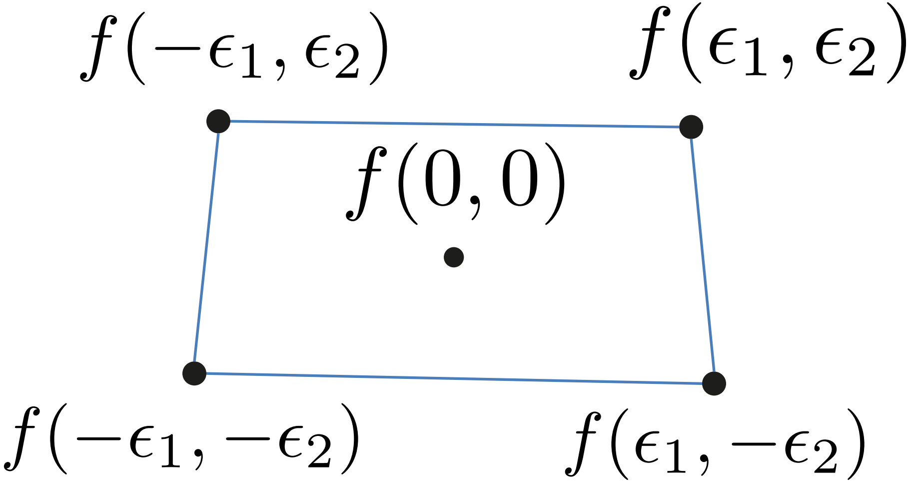
  - Circles stay circles

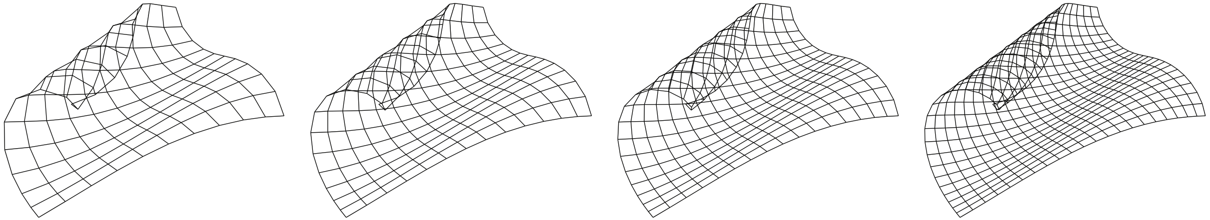


# Infinitesimal quadrilaterals of a smooth curvature line net



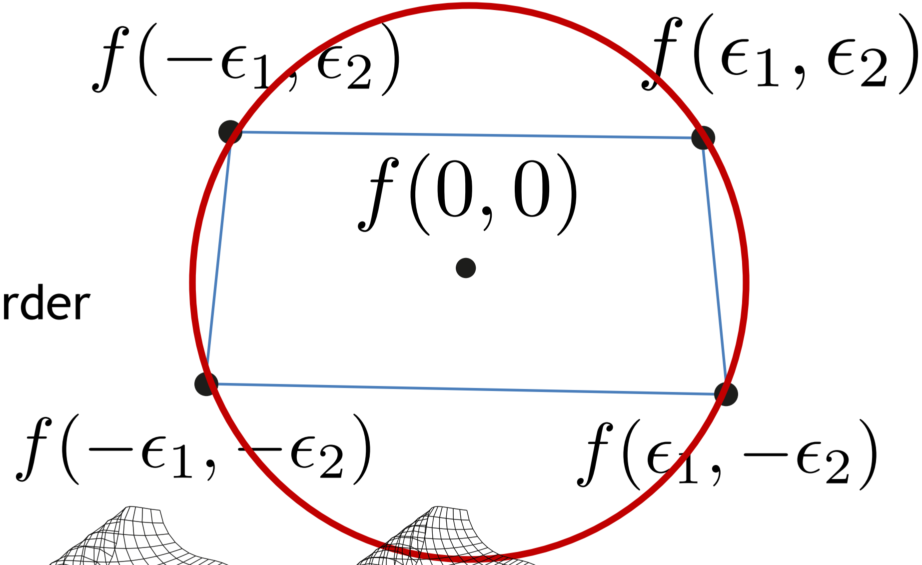
[Alexander I.Bobenko and Ulrich Pinkall 1999]

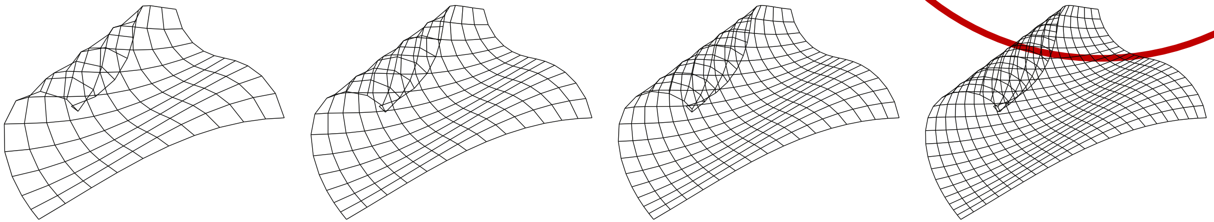
# Infinitesimal quadrilaterals of a smooth curvature line net

A smooth net is a curvature line net

⇕

$(\epsilon_1, \epsilon_2)$ quads are circular up to second order

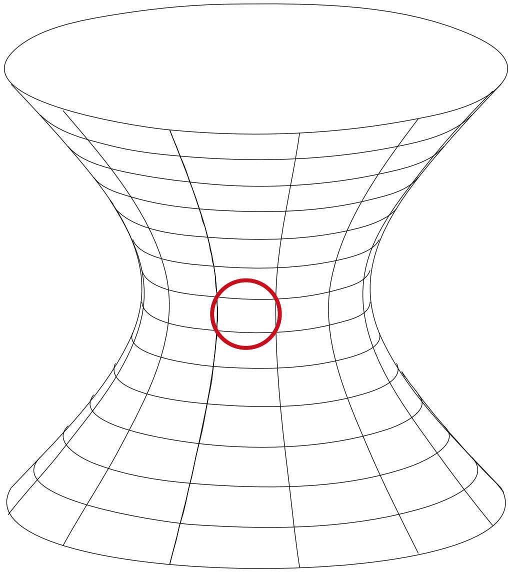


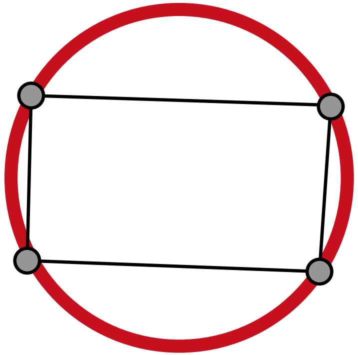



[Alexander I.Bobenko and Ulrich Pinkall 1999]

# Discrete curvature line nets – circular nets

A discrete net with circular faces





Opposite angles sum up to $\pi$

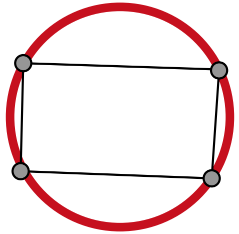
# Curvature line net

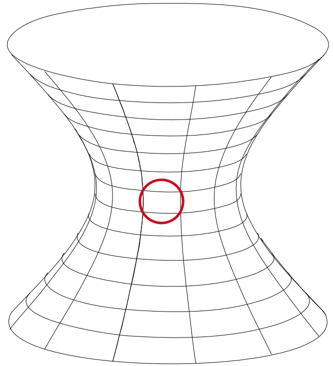
## Smooth

$$n_x \parallel f_x, n_y \parallel f_y$$

## Discrete

Quads are circular

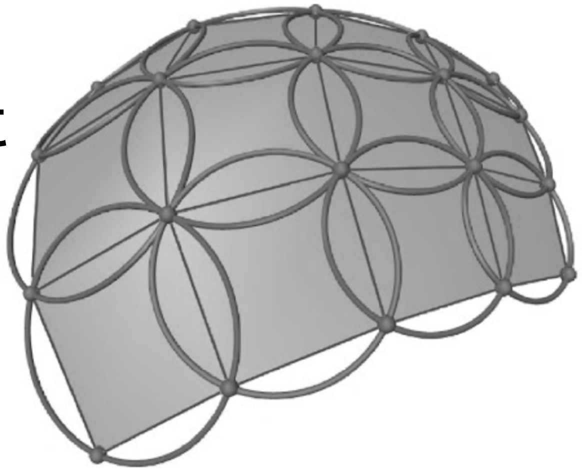




Invariant to Moebius transformations

# Convergence, structure and locking

- Convergence?
- Is there locking?
- Can every smooth curvature line net be discretized by circular nets?

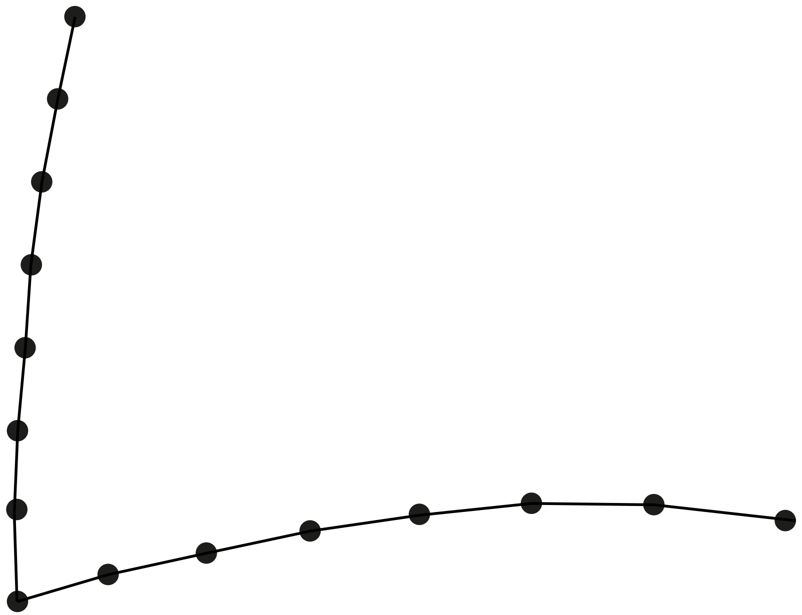


Discrete Differential Geometry :Integrable Structure
[Bobenko and Suris 2008]

# Structure through evolution

Pass a circular net through these curves

$\mathbb{R}^3$



[Alexander I.Bobenko and Yuri B.Suris 2008]

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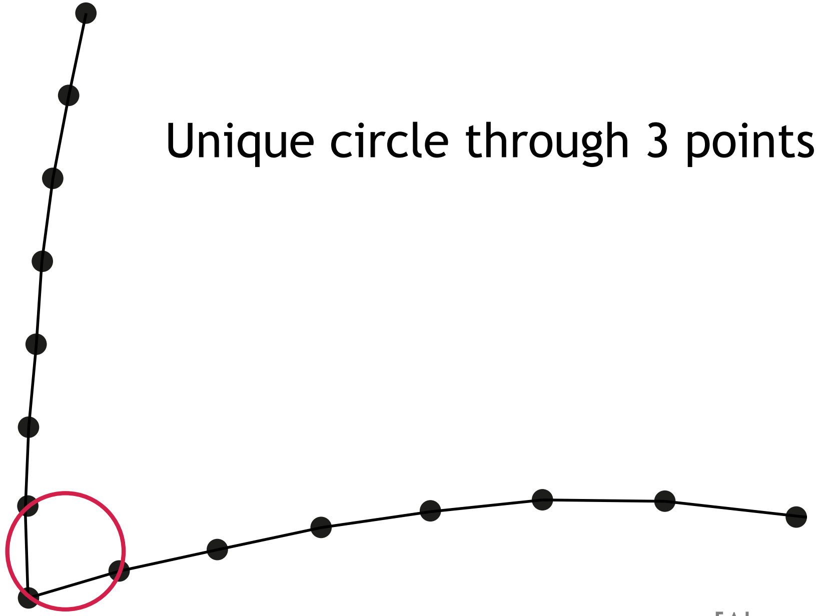


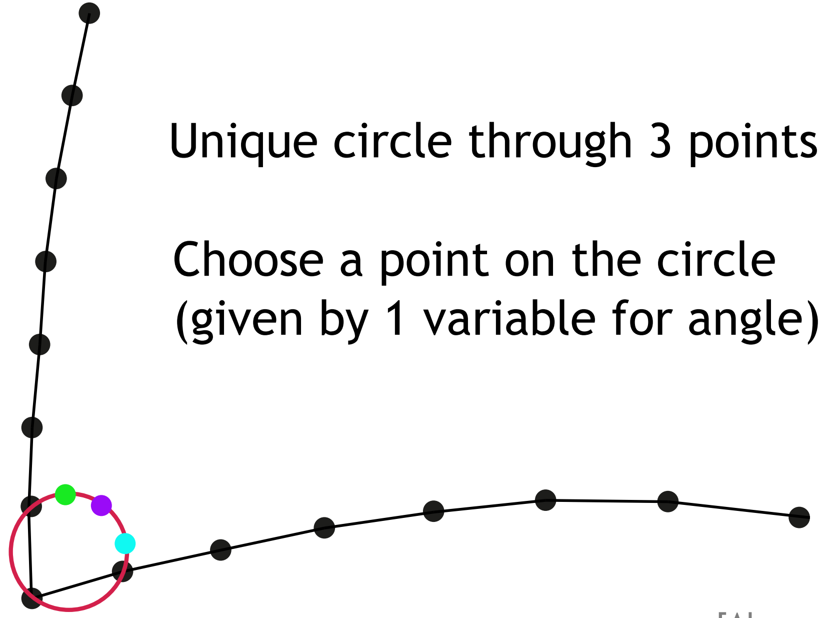
Unique circle through 3 points

[Alexander I.Bobenko and Yuri B.Suris 2008]

# Rigidity through evolution

Pass a circular net through these curves

$\mathbb{R}^3$



Unique circle through 3 points

Choose a point on the circle
(given by 1 variable for angle)

[Alexander I.Bobenko and Yuri B.Suris 2008]

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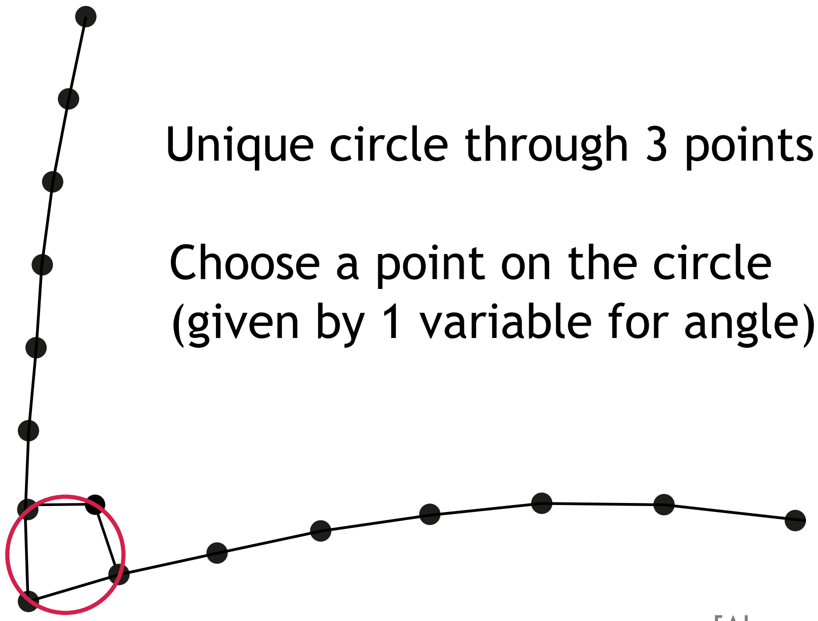
$\mathbb{R}^3$



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[Alexander I.Bobenko and Yuri B.Suris 2008]

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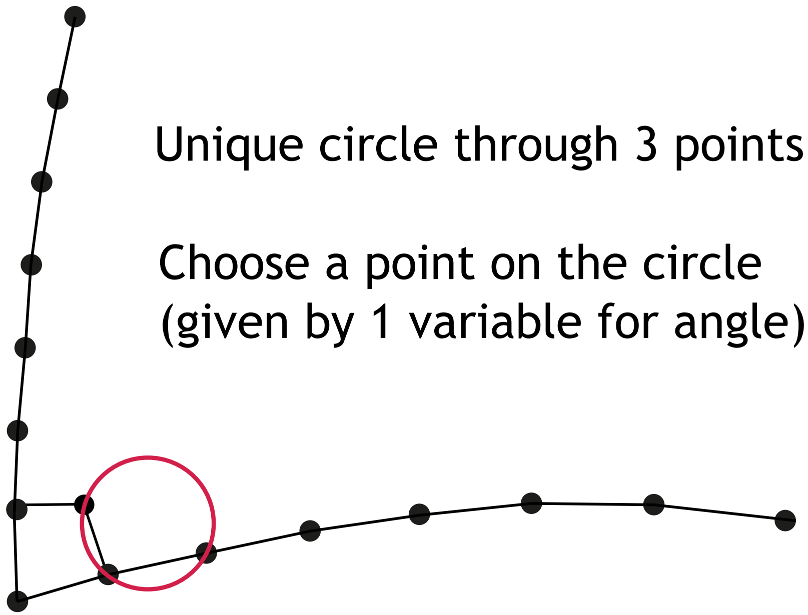

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[Alexander I.Bobenko and Yuri B.Suris 2008]

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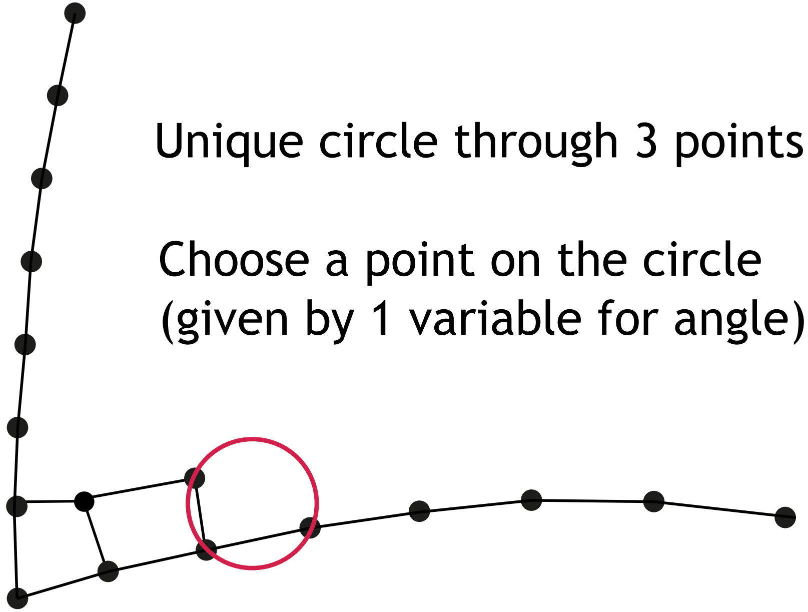

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[Alexander I.Bobenko and Yuri B.Suris 2008]

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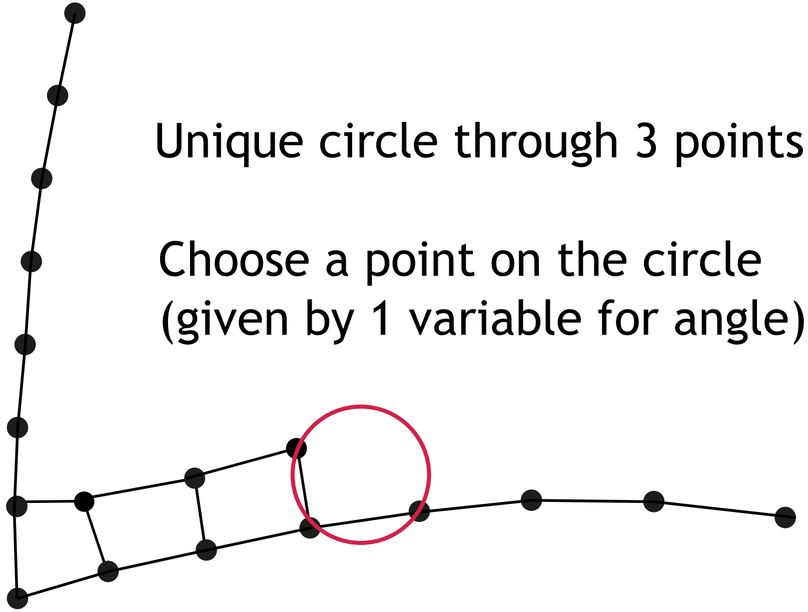

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[Alexander I.Bobenko and Yuri B.Suris 2008]

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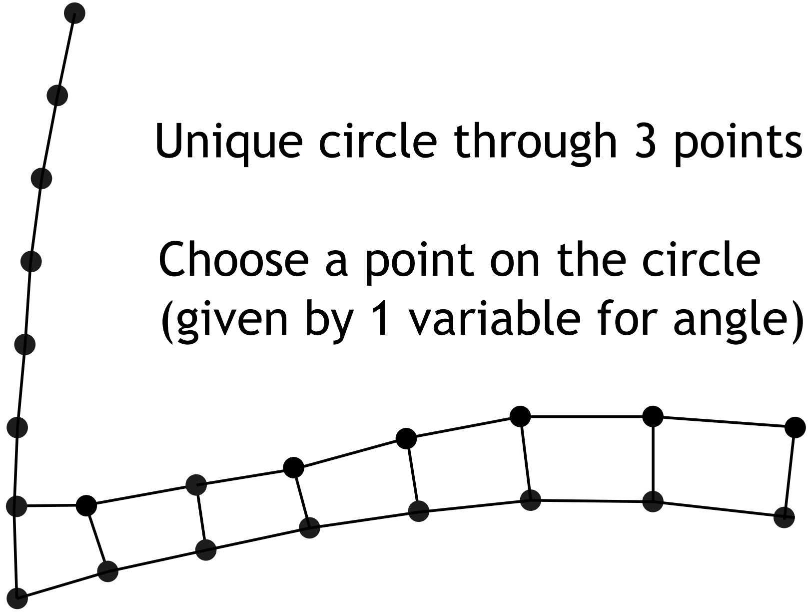

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[Alexander I.Bobenko and Yuri B.Suris 2008]

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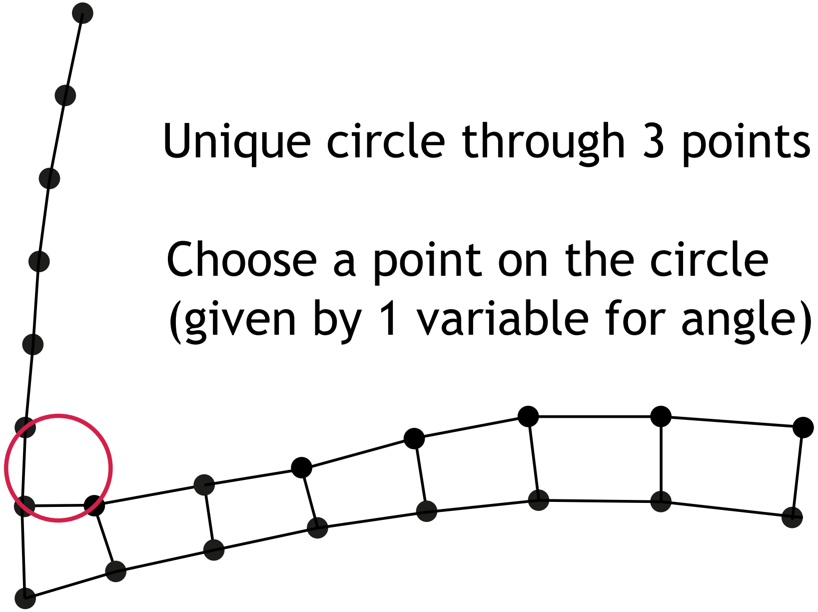
Pass a circular net through these curves

$\mathbb{R}^3$



Unique circle through 3 points

Choose a point on the circle
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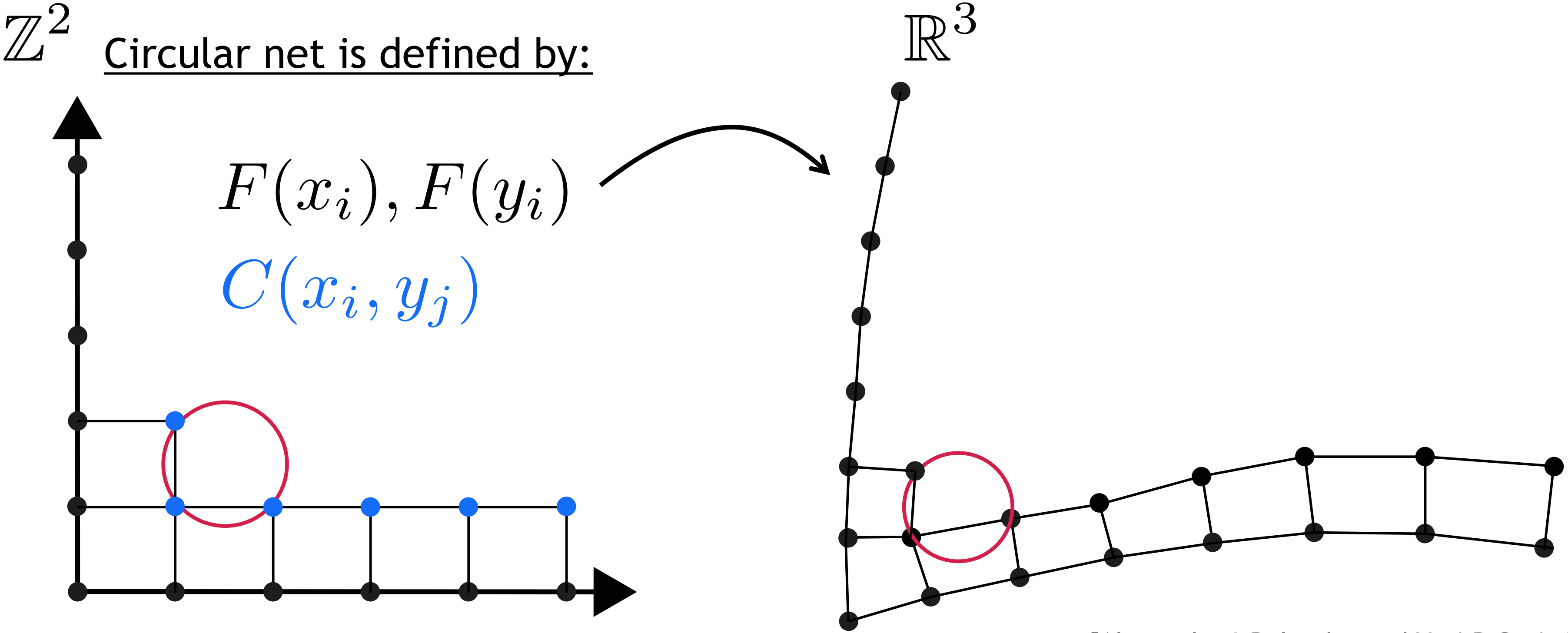
[Alexander I.Bobenko and Yuri B.Suris 2008]

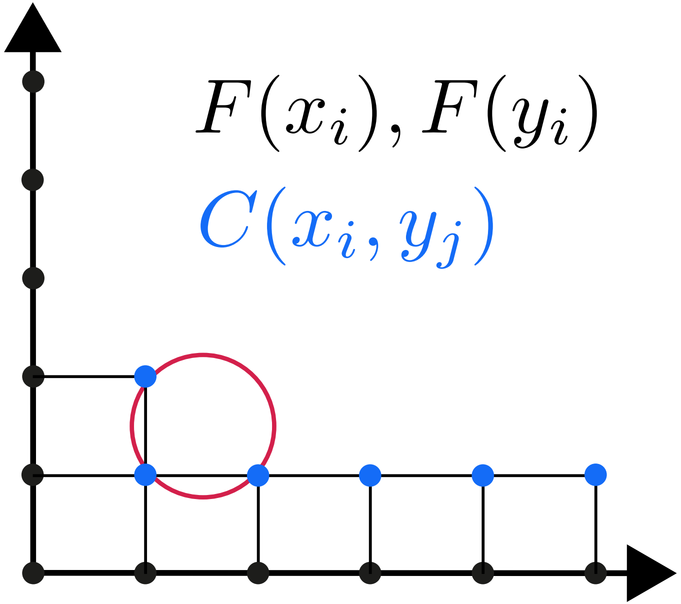
# Rigidity through evolution



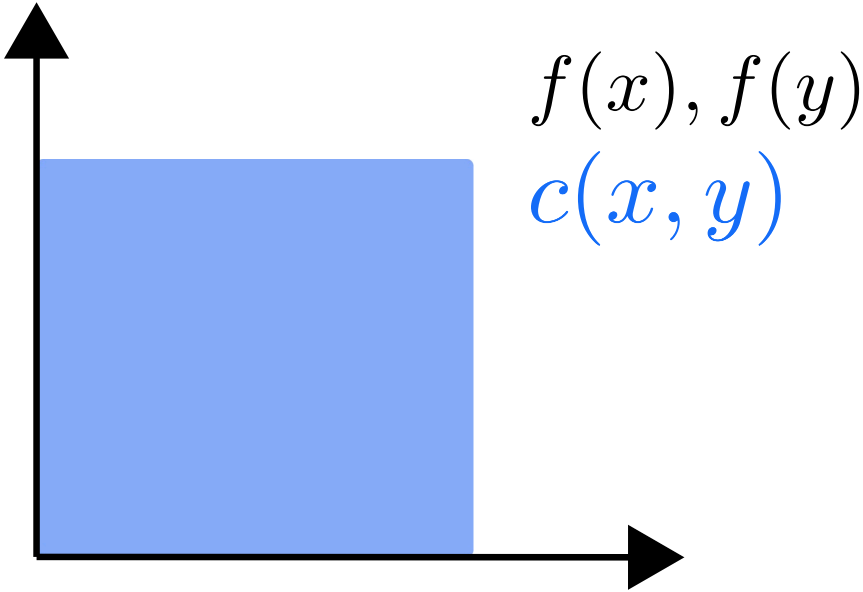

[Alexander I.Bobenko and Yuri B.Suris 2008]

# Smooth evolution

Circular net is defined by:



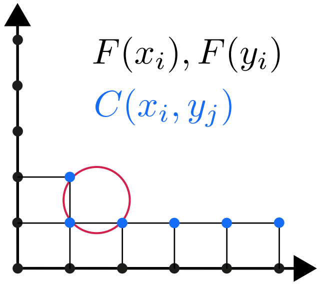

Smooth curvature line net is defined by:

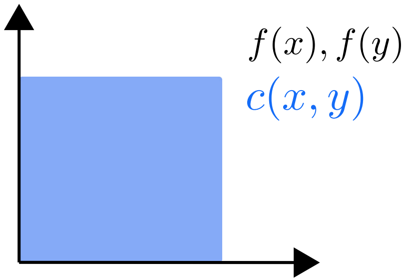



[Alexander I.Bobenko and Yuri B.Suris 2008]

# Evolution applications

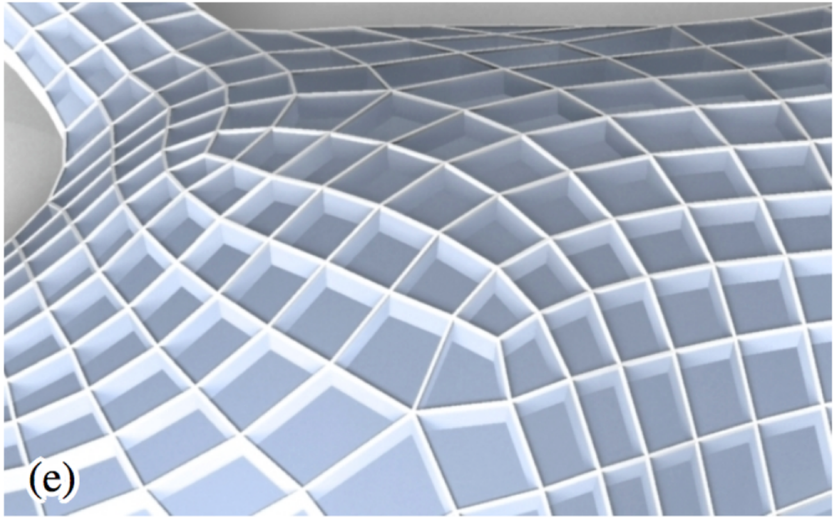
- Convergence
- Structure and no locking
- Optimization
  - Existence of solutions for non linear problems
  - Constraints independence for second order methods



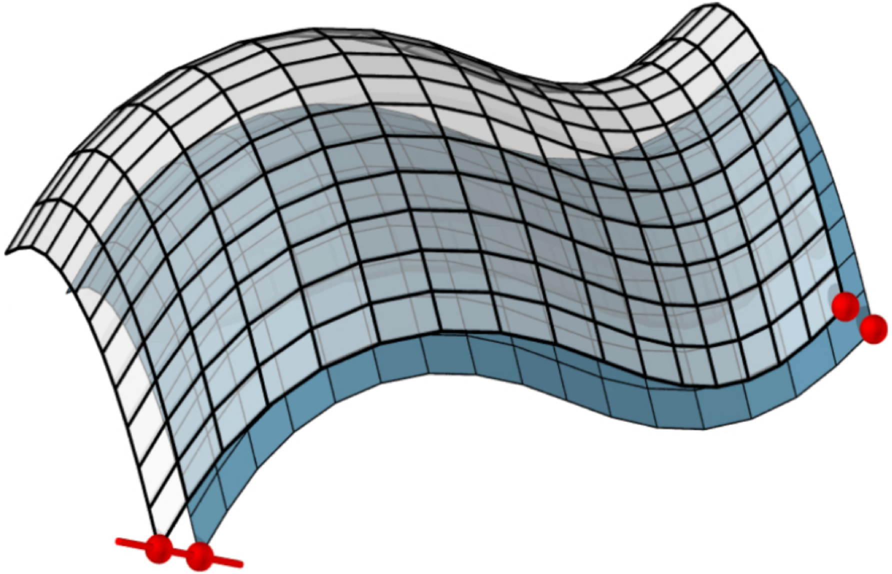



# Circular meshes

Support structures



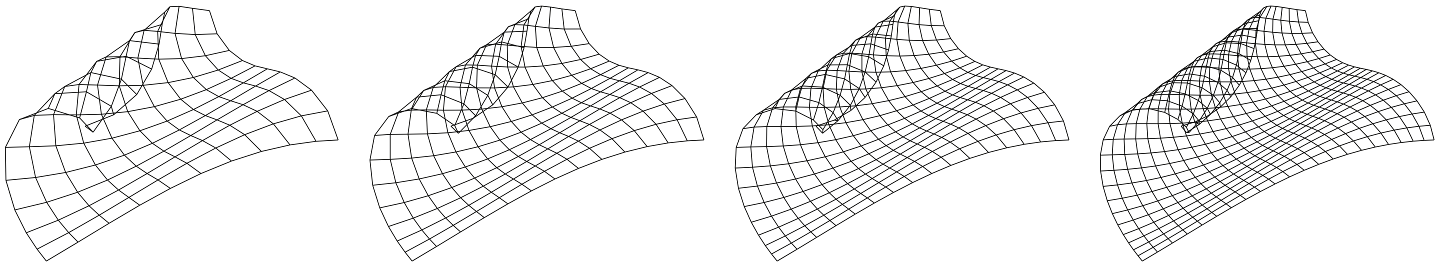

Parallel meshes



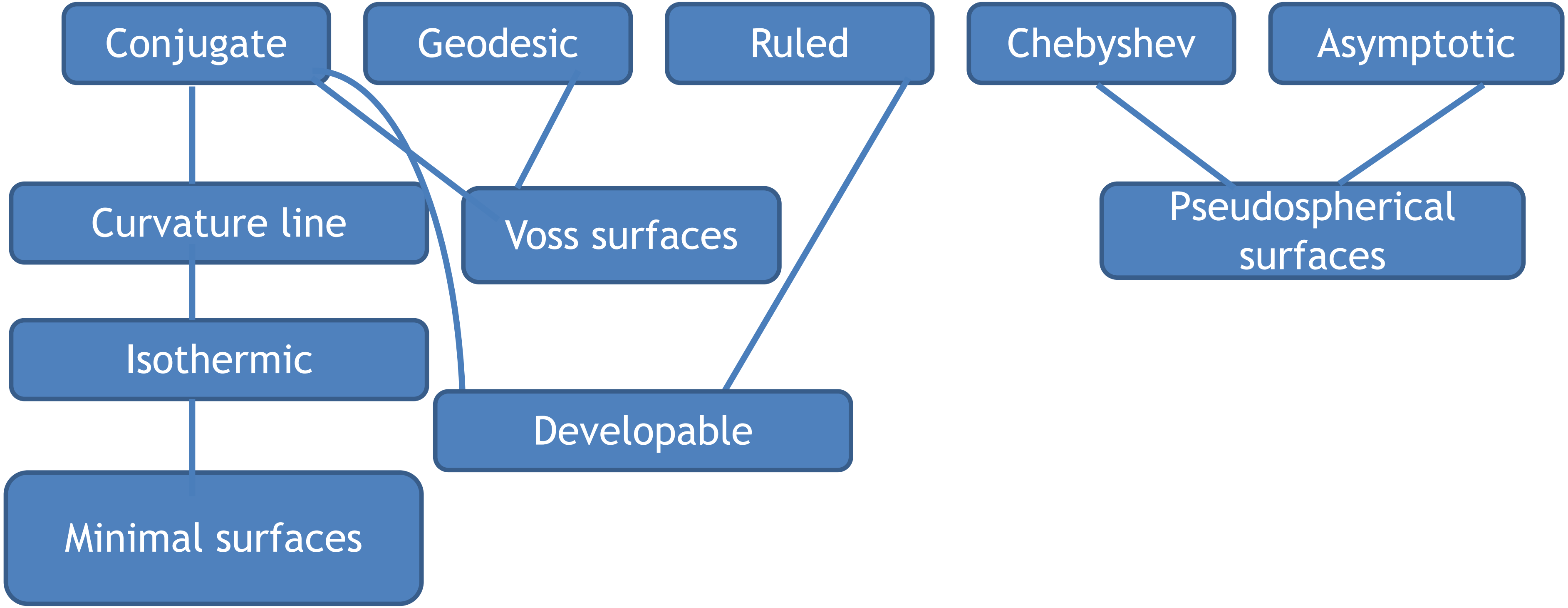
[Pottmann et al. 2007]

# Properties of a good discretization

- Converges to the smooth counterpart
- Simple
- Preserves structure

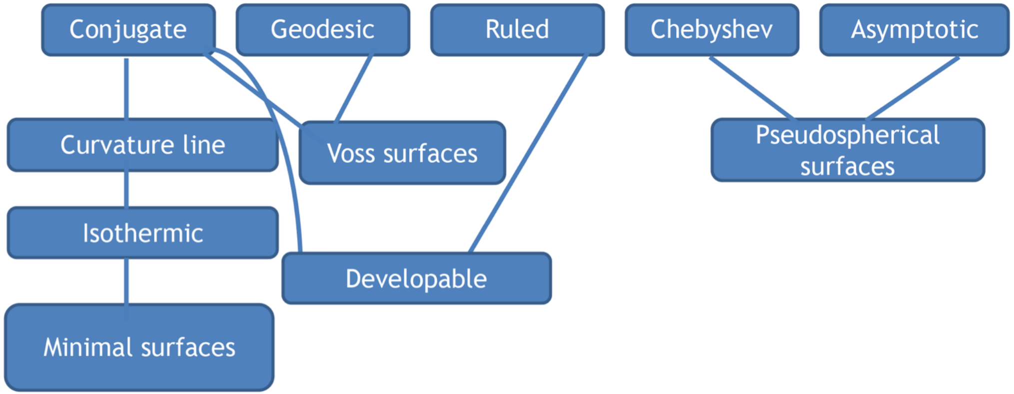


# A zoo of smooth nets



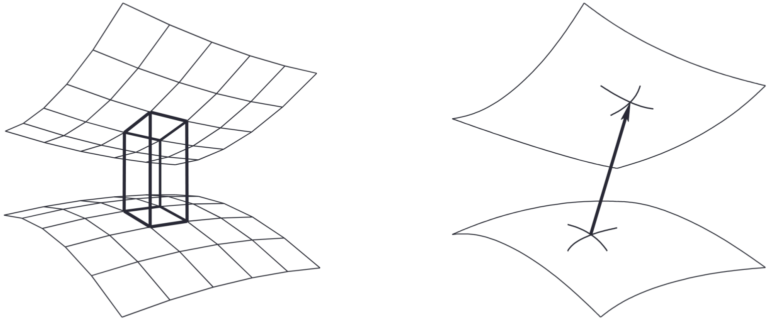
# A rich discrete theory

- Simpler definitions replacing differential properties
  - Angles
  - Length
  - Planarity
  - Intersection



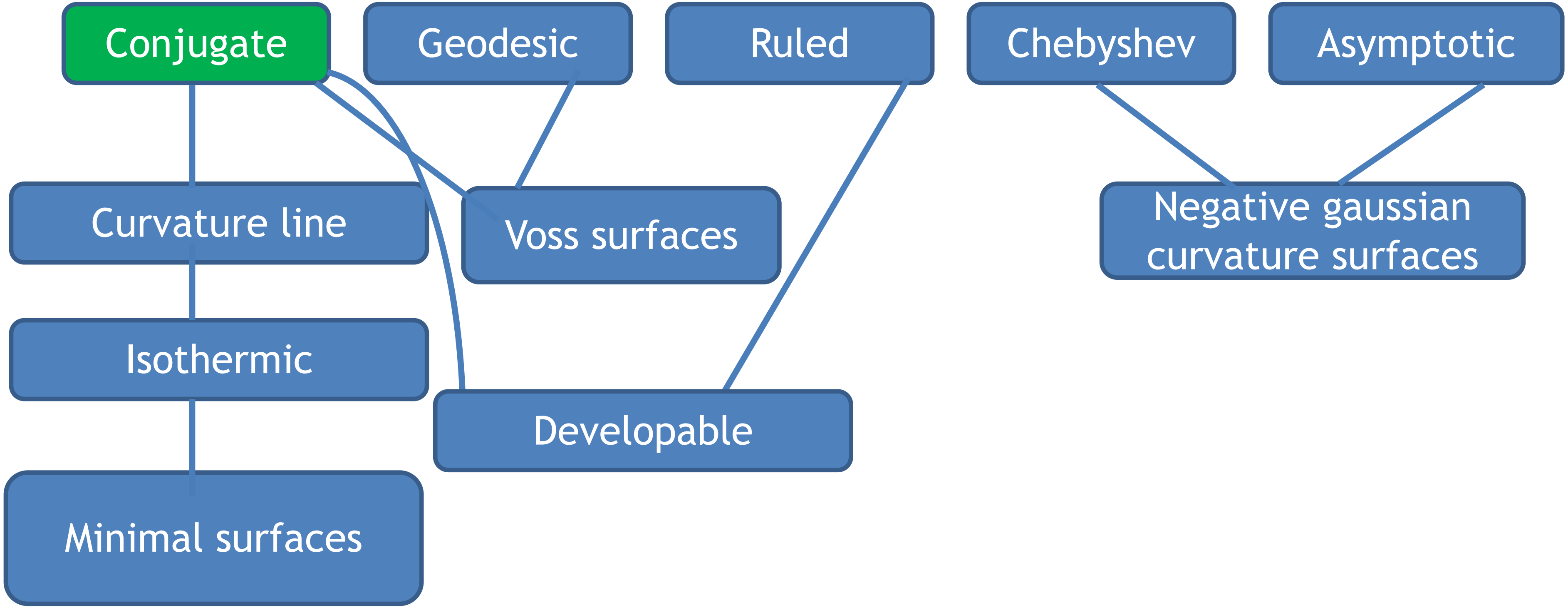
# Applications of discrete nets

- Theoretical
  - Smooth geometry
  - Mathematical physics
- Modeling and optimization



Discrete Differential Geometry :Integrable Structure
[Bobenko and Suris 2008]

# A zoo of discrete nets



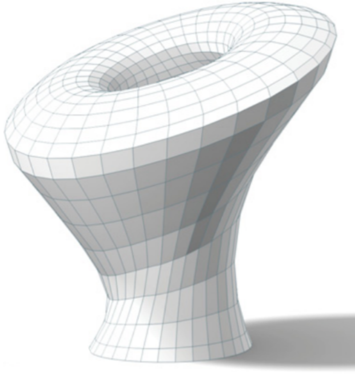
# Conjugate nets

**Smooth**

$$\langle f_x, n_y \rangle = 0$$

**Discrete**

Planar quad meshes



[Bouaziz et. al. 2012]

Invariant to projective transformations
Curvature lines are conjugate

# Conjugate nets

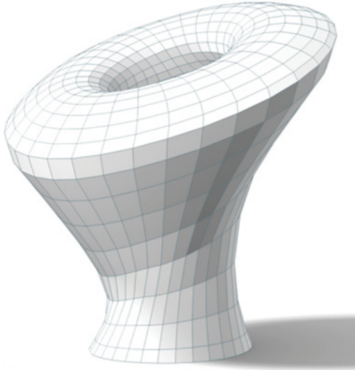
## Smooth

$$\langle f_x , n_y \rangle = 0$$

???

## Discrete

Planar quad meshes



Invariant to projective transformations

Curvature lines are conjugate

[Bouaziz et. al. 2012]

# Conjugate nets derivation

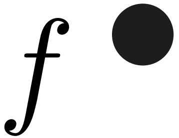
$$\langle f_x, n\rangle = 0 \Rightarrow \langle f_x, n\rangle_y = 0$$

$$\Rightarrow \langle f_{xy}, n\rangle + \langle f_x, n_y\rangle = 0$$

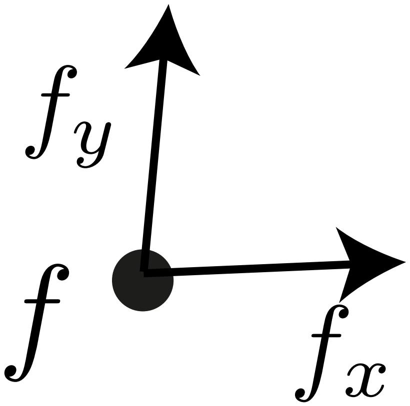
$f_x, f_y$ conjugate directions $\Leftrightarrow$ $\langle f_x, n_y\rangle = 0$

$$\Leftrightarrow \langle f_{xy}, n\rangle = 0 \Leftrightarrow f_{xy} \in span\{f_x, f_y\}$$

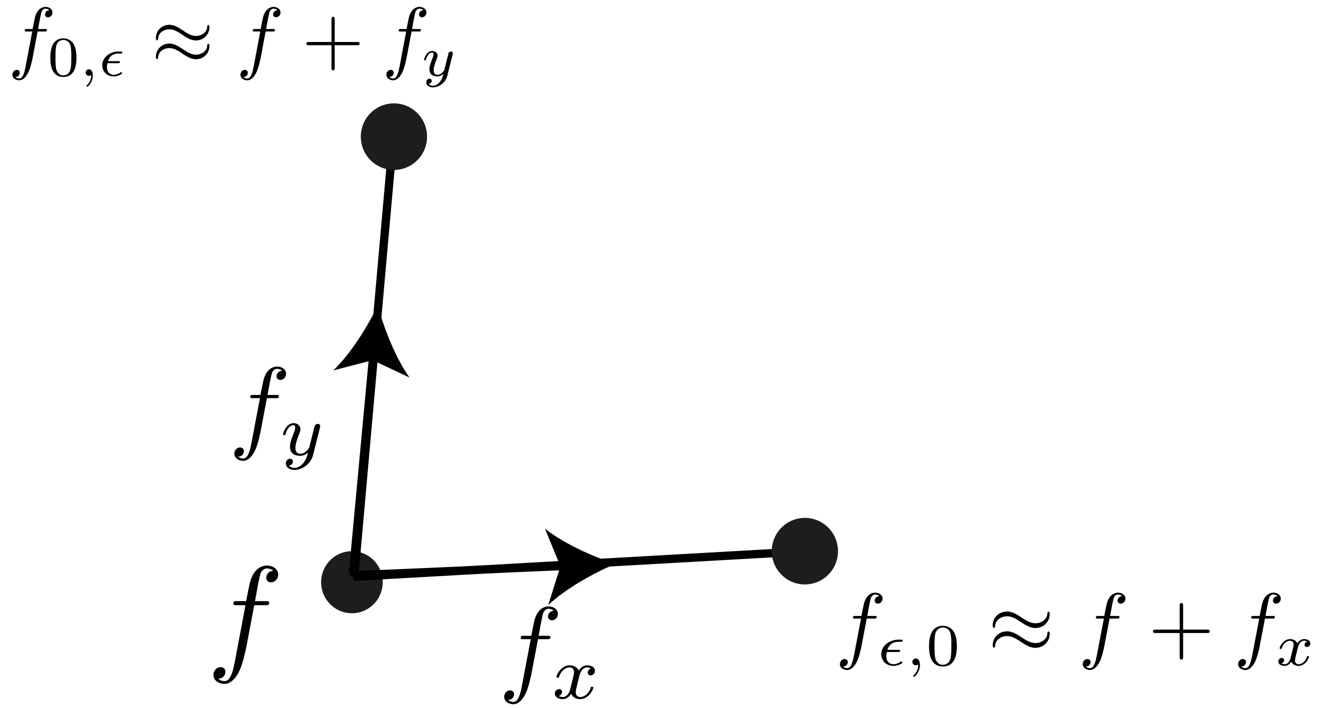
$$f_{xy} \in span\{f_x, f_y\}$$



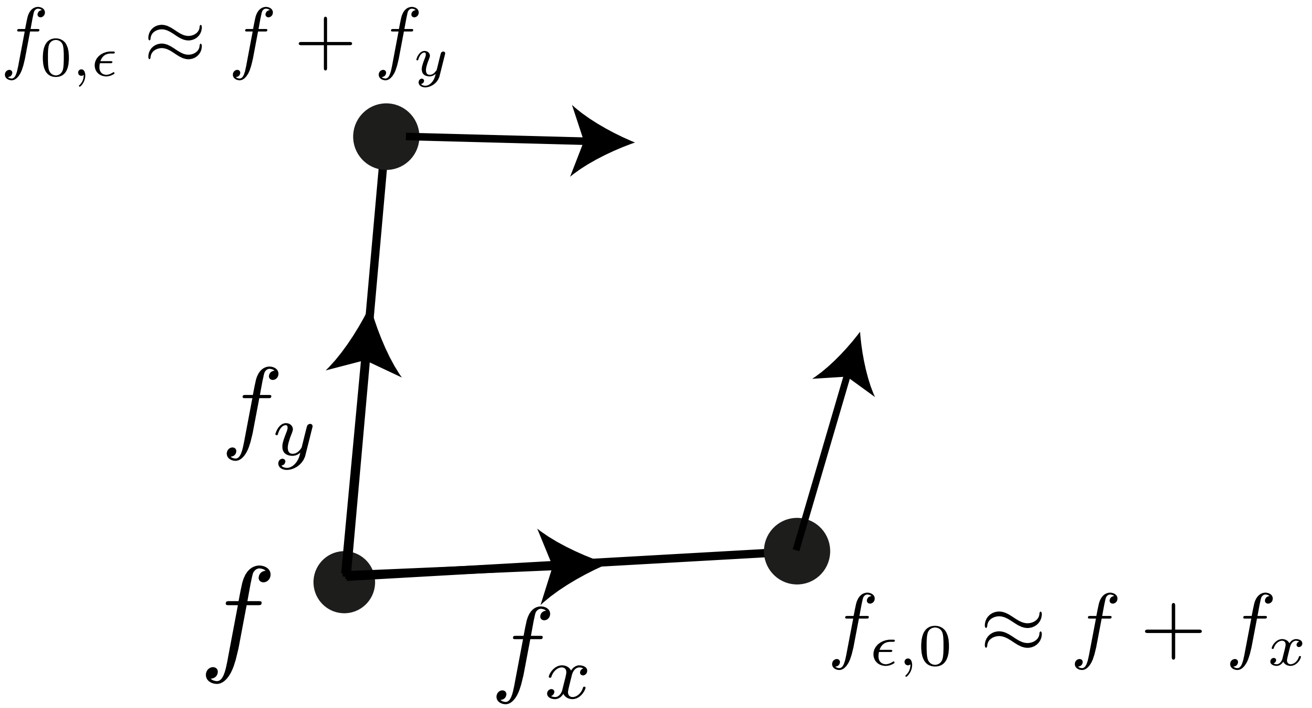
$$f_{xy} \in span\{f_x, f_y\}$$



## $f_{xy} \in span\{f_x, f_y\}$



# $f_{xy} \in span\{f_x, f_y\}$



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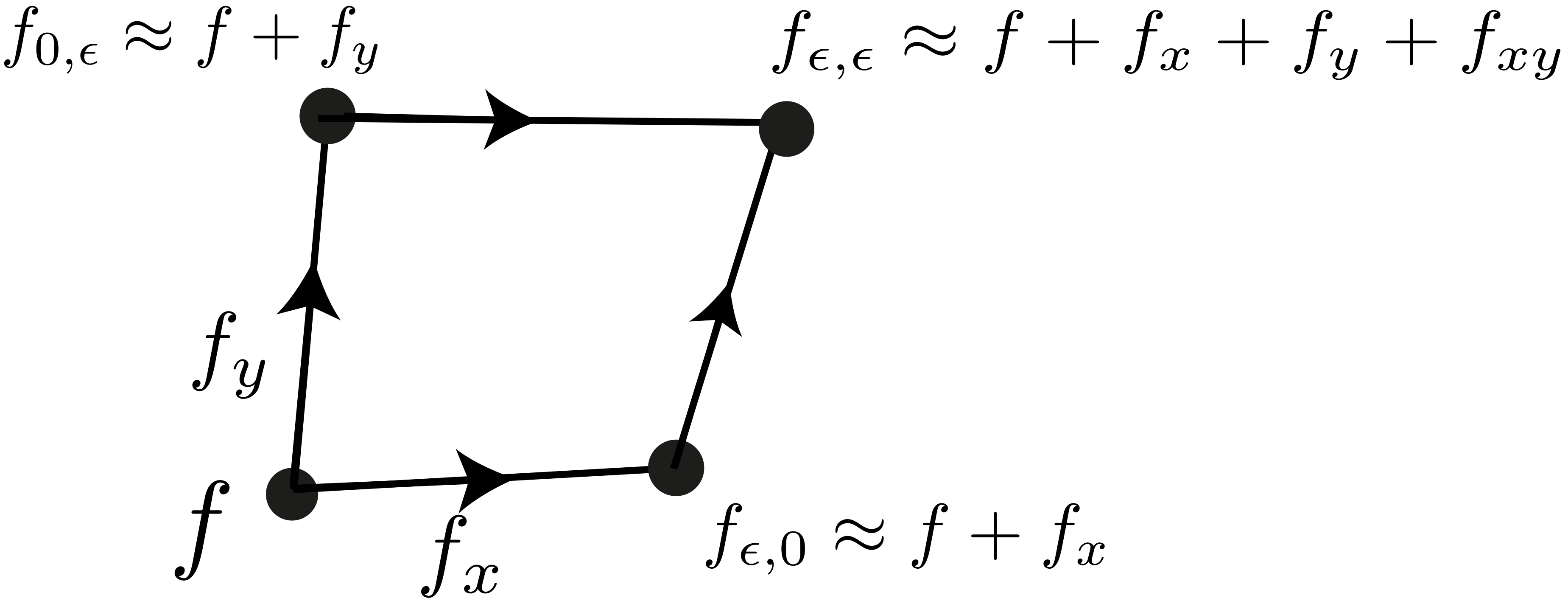
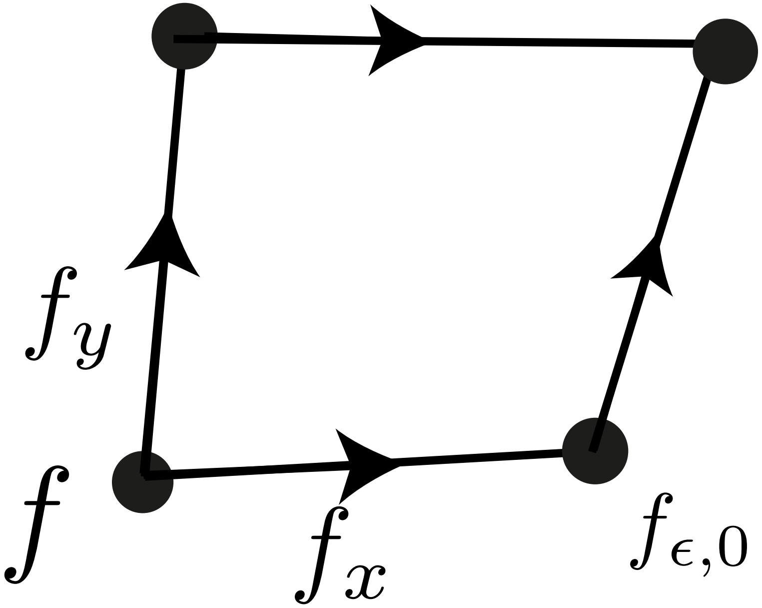


$$f_{xy} \in span\{f_x, f_y\}$$

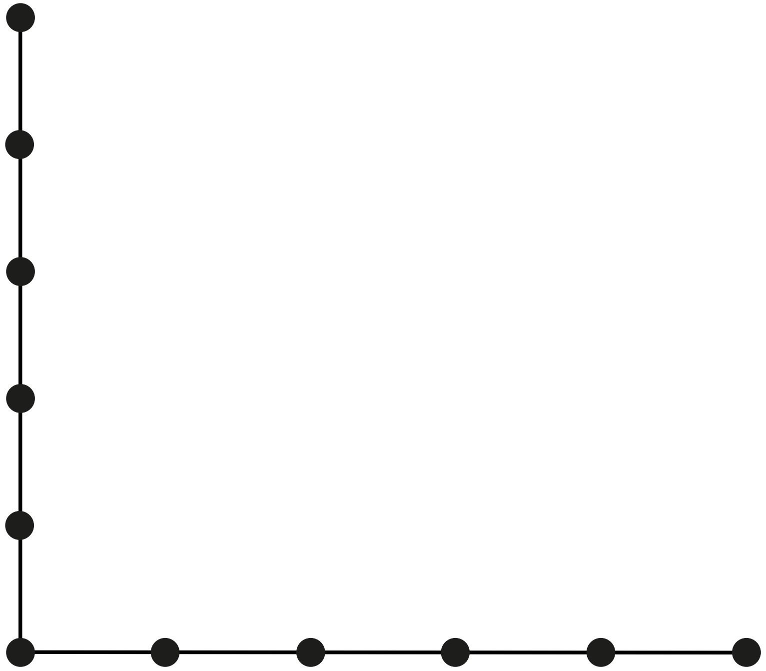
$$f_{0,\epsilon} \approx f + f_y \qquad f_{\epsilon,\epsilon} \approx f + f_x + f_y + f_{xy}$$




$$f_{\epsilon,0} \approx f + f_x$$

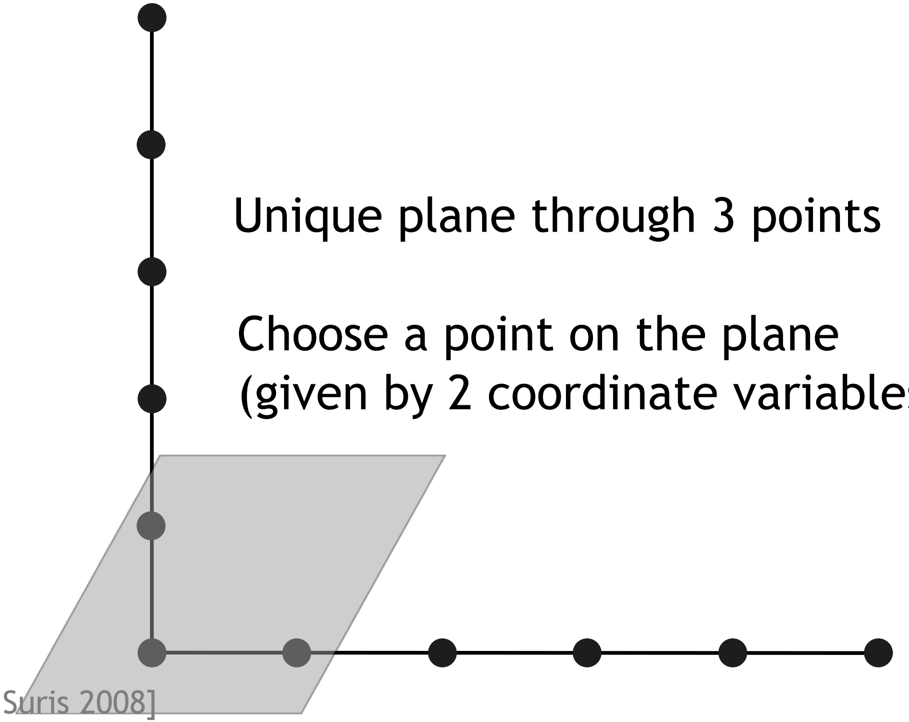
Infinitesimal quads are planar

# Evolution of (discrete) conjugate nets



[Alexander I.Bobenko and Yuri B.Suris 2008]

# Evolution of (discrete) conjugate nets

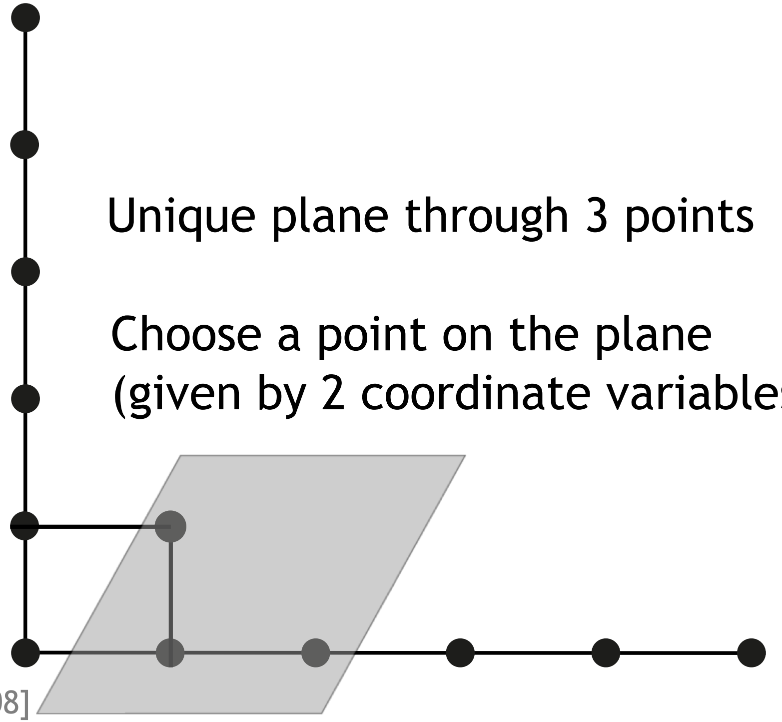


Unique plane through 3 points

Choose a point on the plane
(given by 2 coordinate variables)

[Alexander I.Bobenko and Yuri B.Suris 2008]

# Evolution of (discrete) conjugate nets



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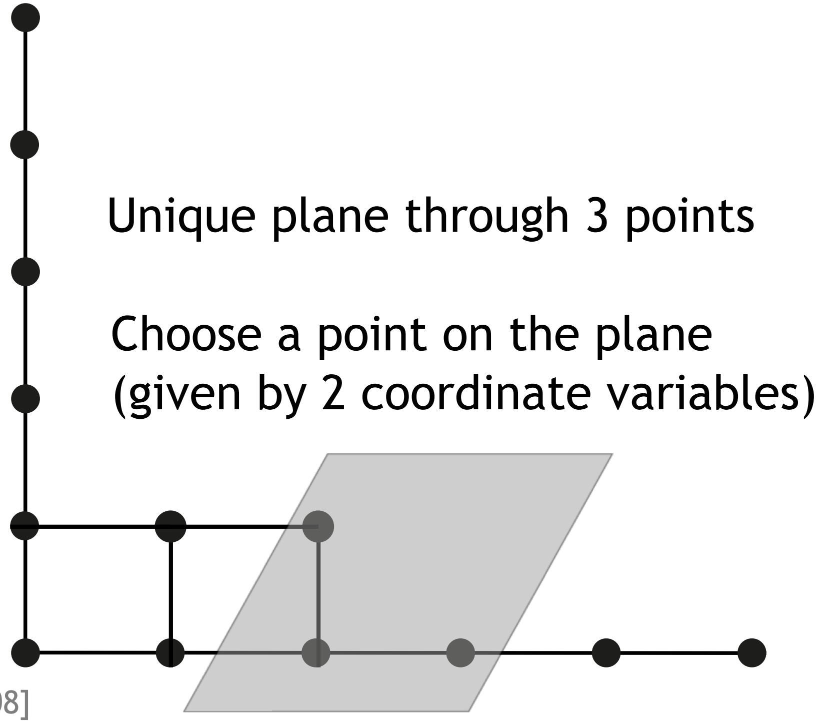


[Alexander I.Bobenko and Yuri B.Suris 2008]

# Evolution of (discrete) conjugate nets

Unique plane through 3 points

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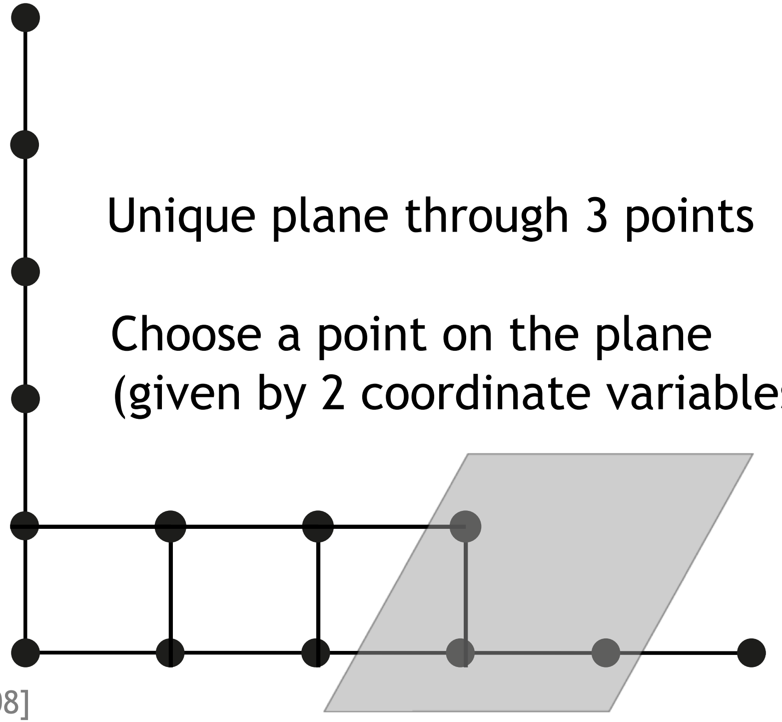

[Alexander I.Bobenko and Yuri B.Suris 2008]

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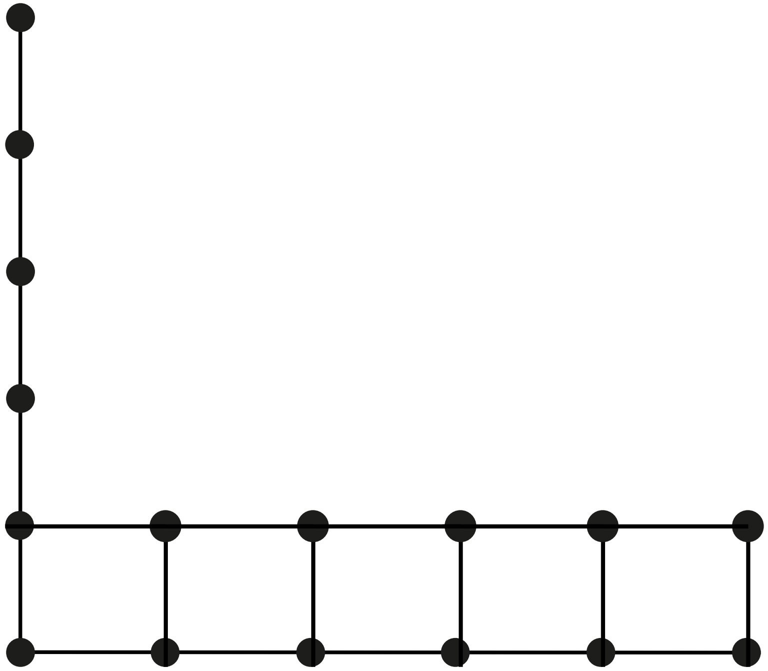
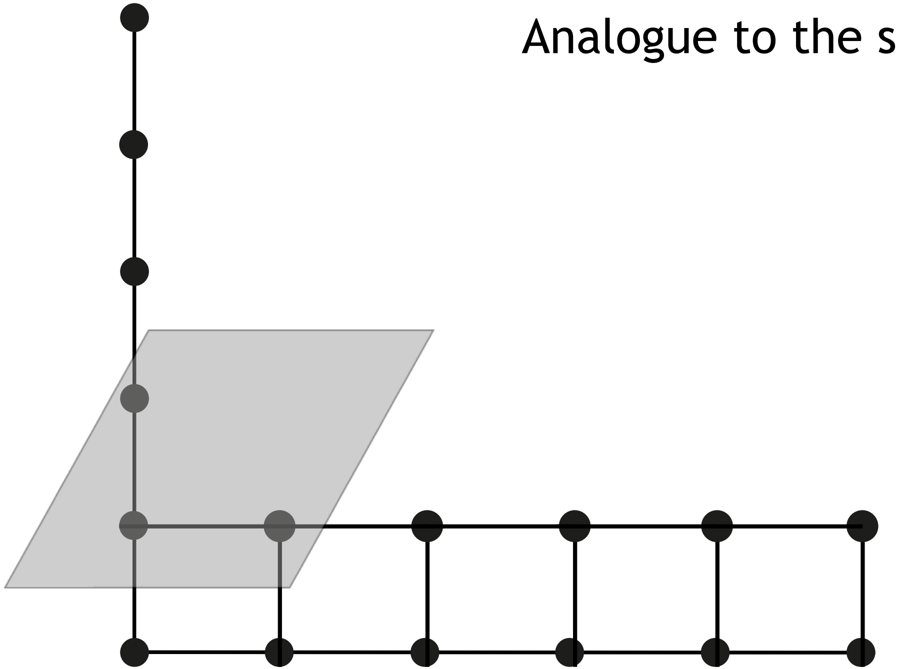


[Alexander I.Bobenko and Yuri B.Suris 2008]

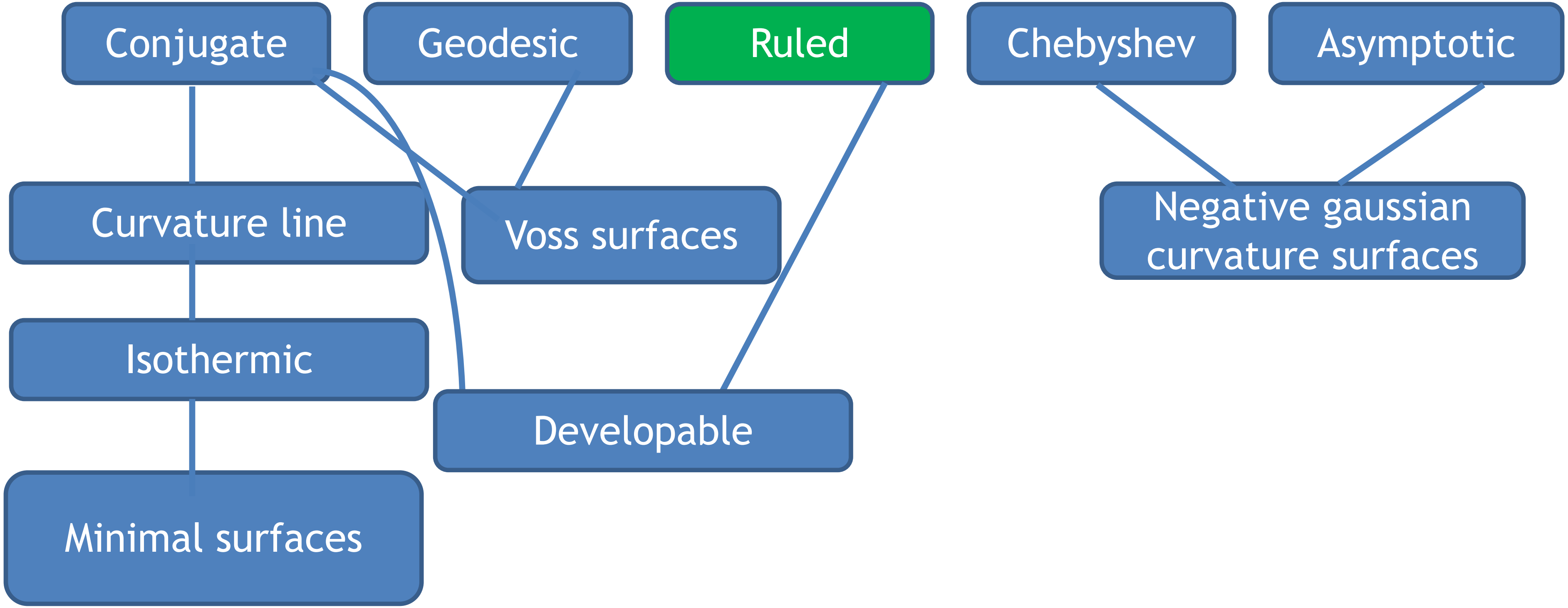
# Evolution of (discrete) conjugate nets

Analogue to the smooth construction

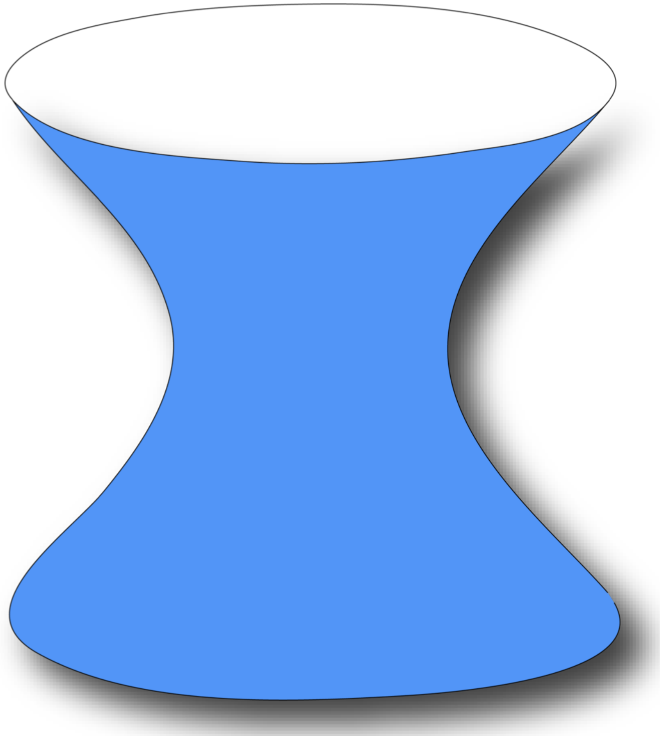


[Alexander I.Bobenko and Yuri B.Suris 2008]

# A zoo of discrete nets



# Ruled Surfaces



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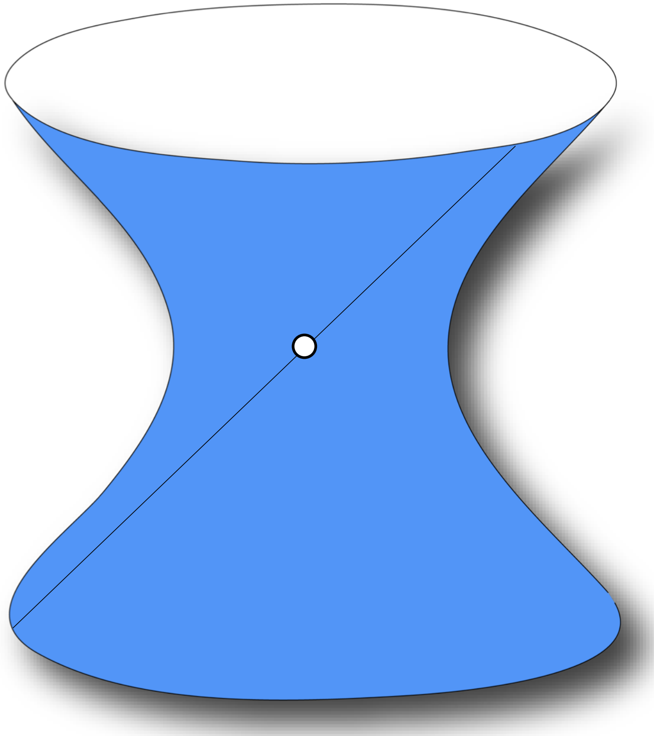

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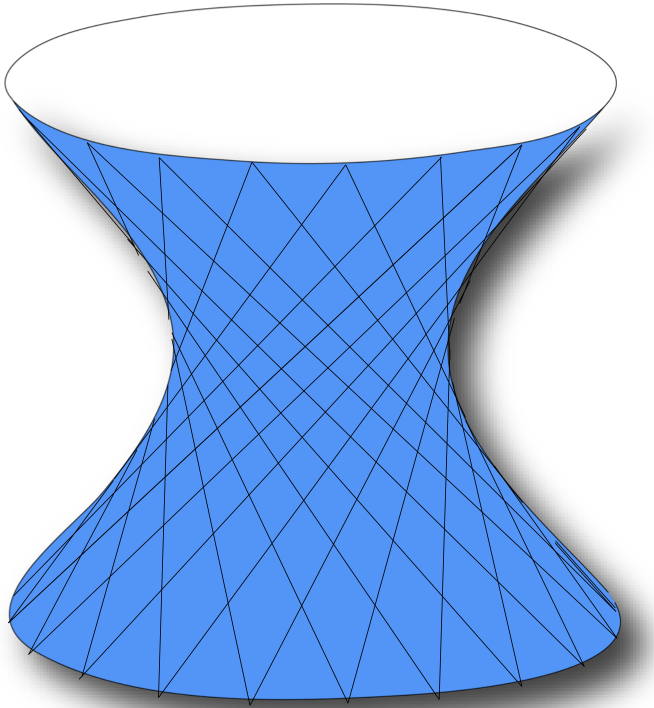


# Ruled Surfaces

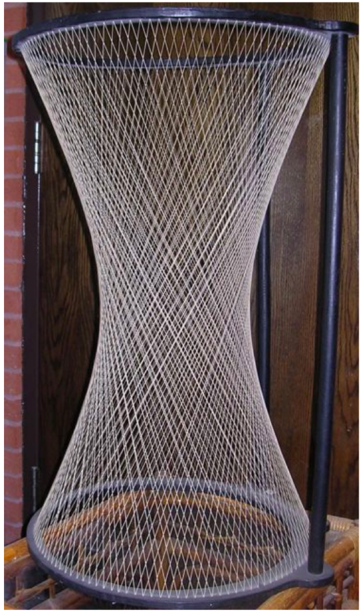

The Warszawa Ochota railway station,
Warsaw, Poland, 1962.

The Ciechanów water tower,
Ciechanów, Poland

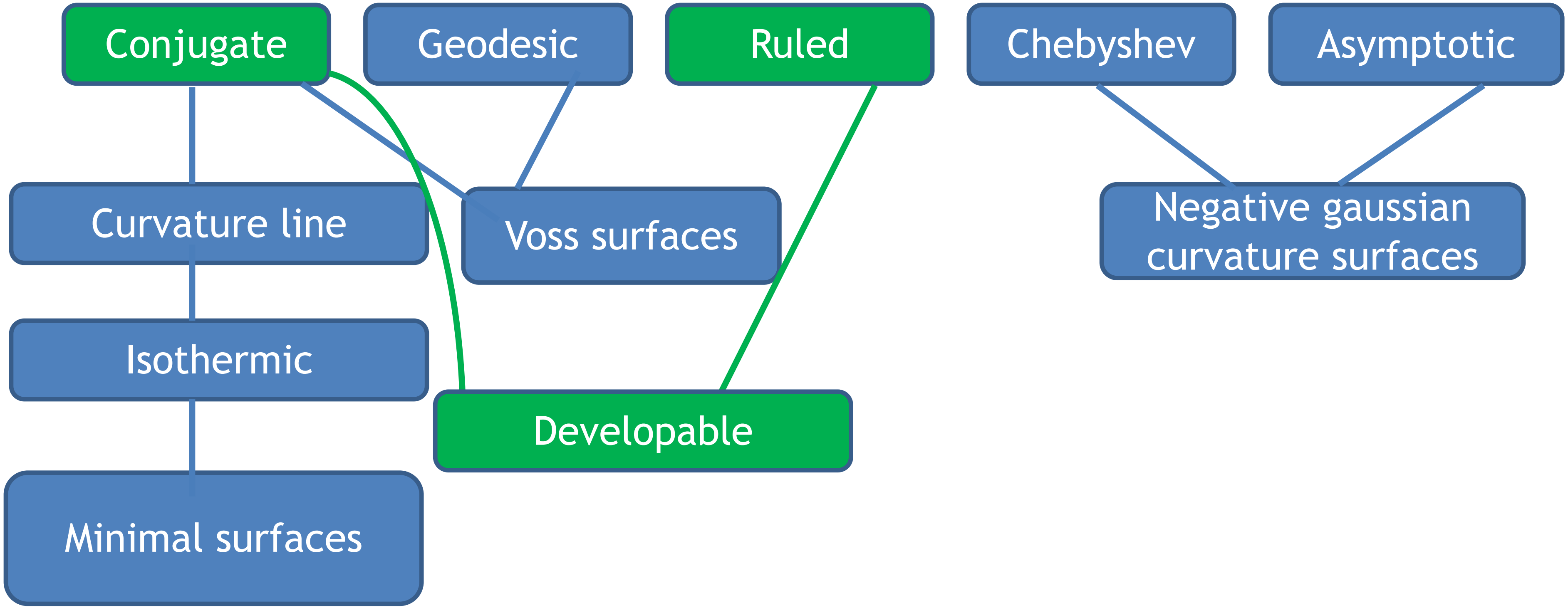
Saint Louis Science Center,
St. Louis, Missouri, United States

# Ruled Surfaces

- Always with Gaussian curvature <= 0
- Trivial to discretize
  - Ruling parameterization generally not conjugate
    - Quads non-planar

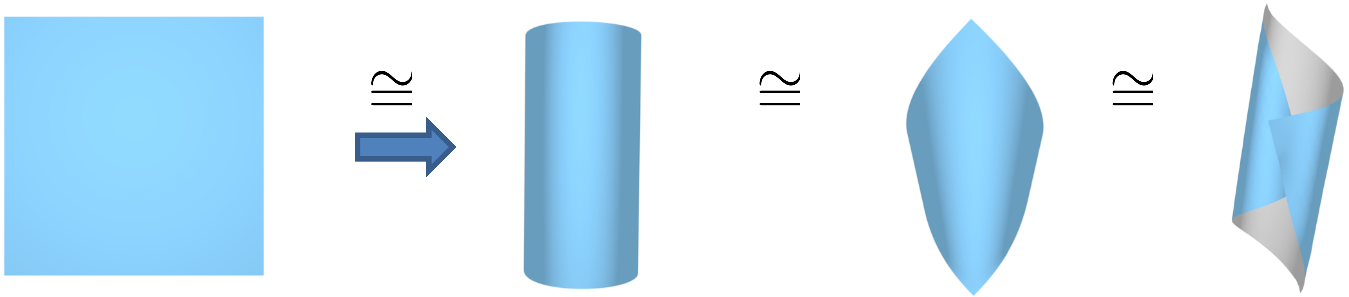


# A zoo of discrete nets



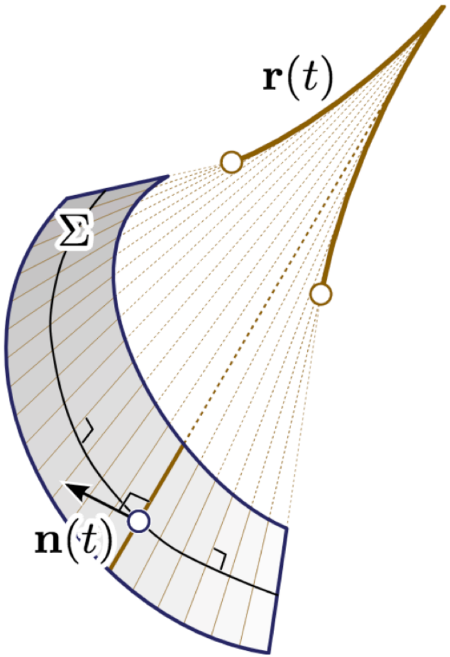
# Conjugate ruled net

- Conjugate + Ruled ⇨ Developable
  - Locally isometric to a planar surface
  - Constant normal along rulings
  - Invariant to projections



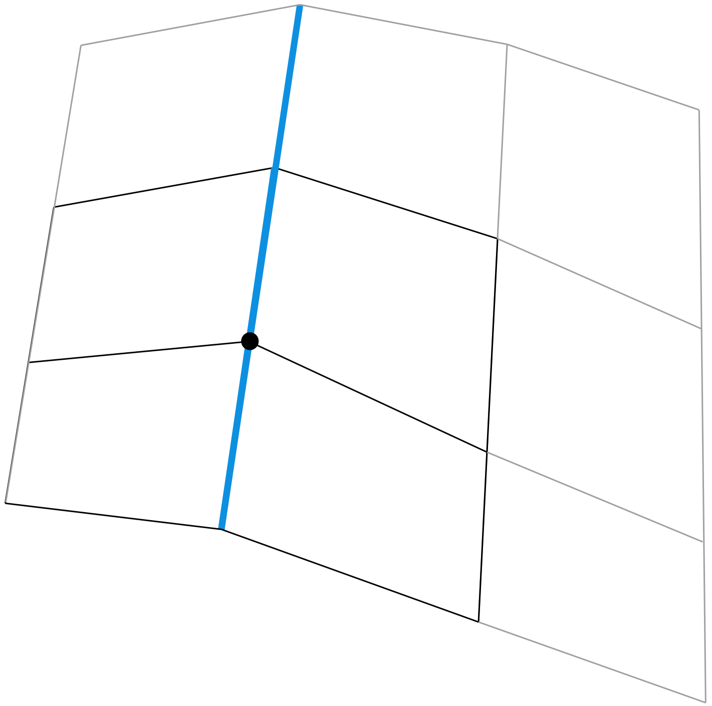
# Discrete conjugate ruled

- Ruled surface with planar quads
  - Locally isometric to a planar surface
  - Constant normal along rulings
  - Invariant to projections

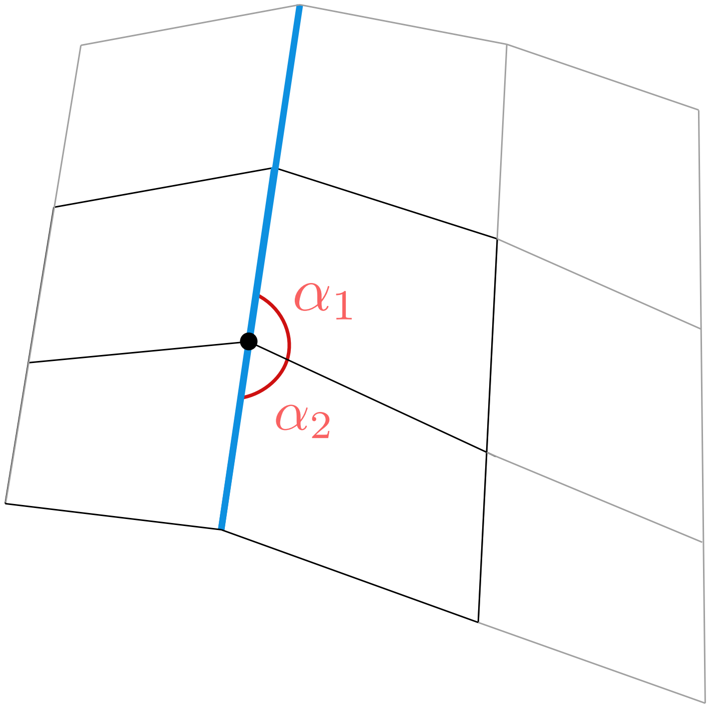



[Sauer 1970]
[Pottmann and Wallner 2001]
[Liu et al. 2007]

# Discrete Conjugate Ruled



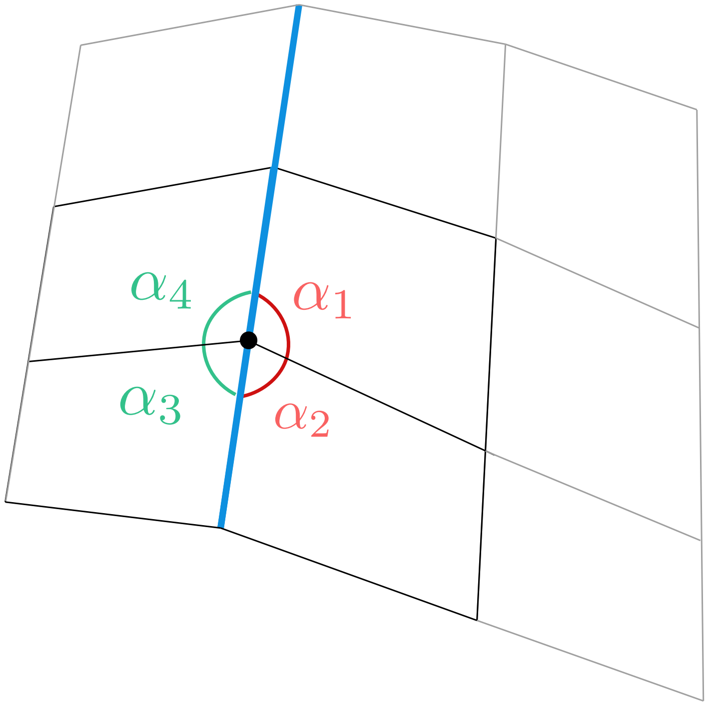
# Discrete Conjugate Ruled




Supplementary angles along a straight line

$$\alpha_1 + \alpha_2 = \pi$$

# Discrete Conjugate Ruled



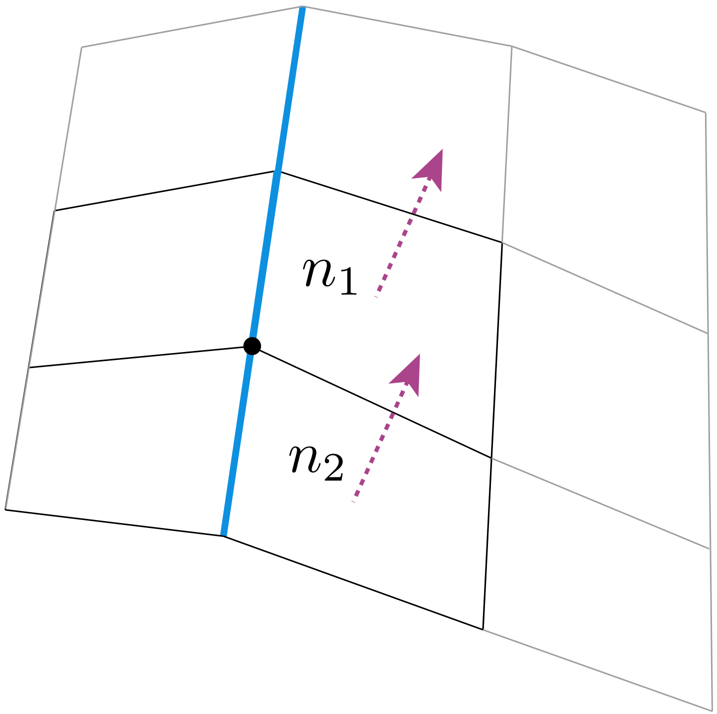

<u>Supplementary angles along a straight line</u>

$$\alpha_1 + \alpha_2 = \pi$$

$$\alpha_3 + \alpha_4 = \pi$$

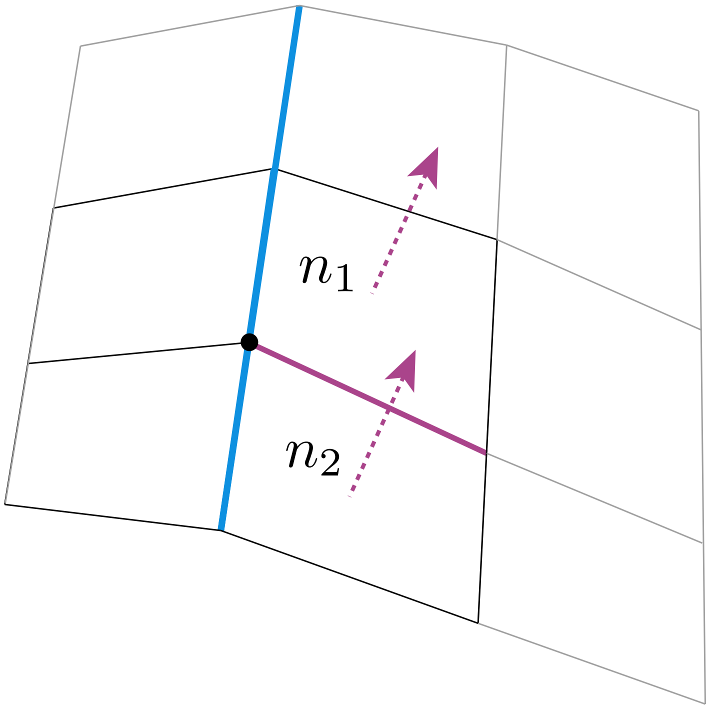
Angles around vertex sum to $2\pi$
Locally isometric to the plane

# Discrete Conjugate Ruled




$$n_1 \parallel n_2$$

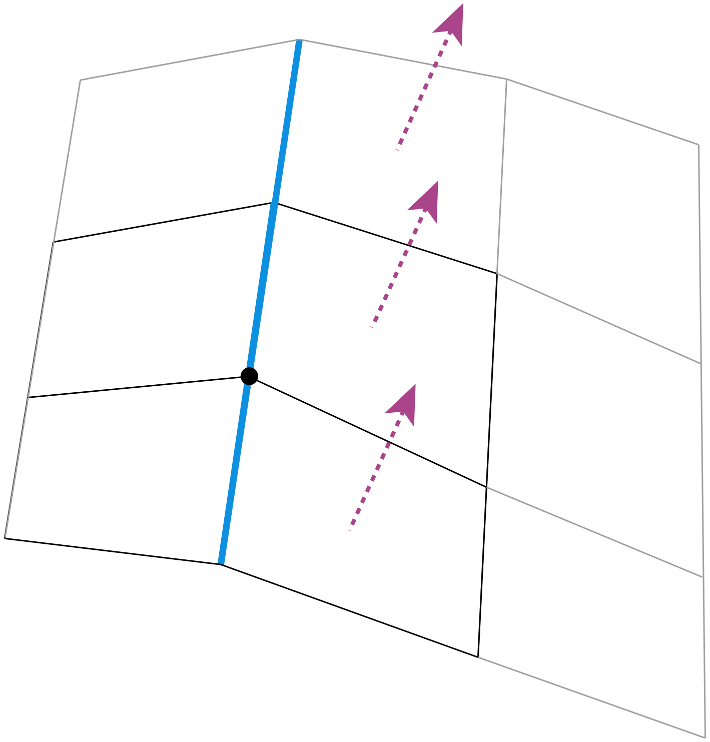
# Discrete Conjugate Ruled




$$n_1 \parallel n_2$$

Orthogonal to the blue ruling line and the purple common edge

# Discrete Conjugate Ruled

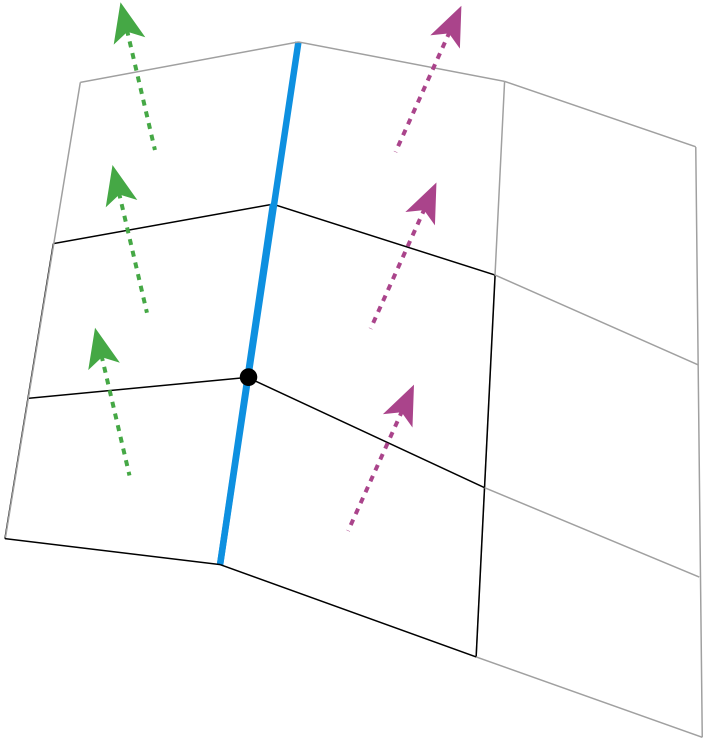


$$n_1 \parallel n_2$$

Orthogonal to the blue ruling line and the purple common edge

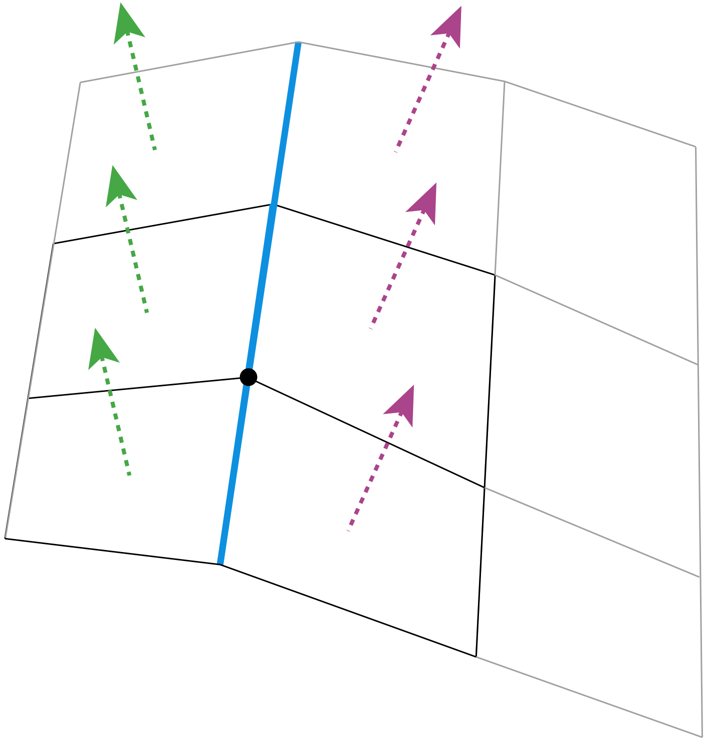
Propagates “up” along the ruling

# Discrete Conjugate Ruled



Constant normal along the rulings

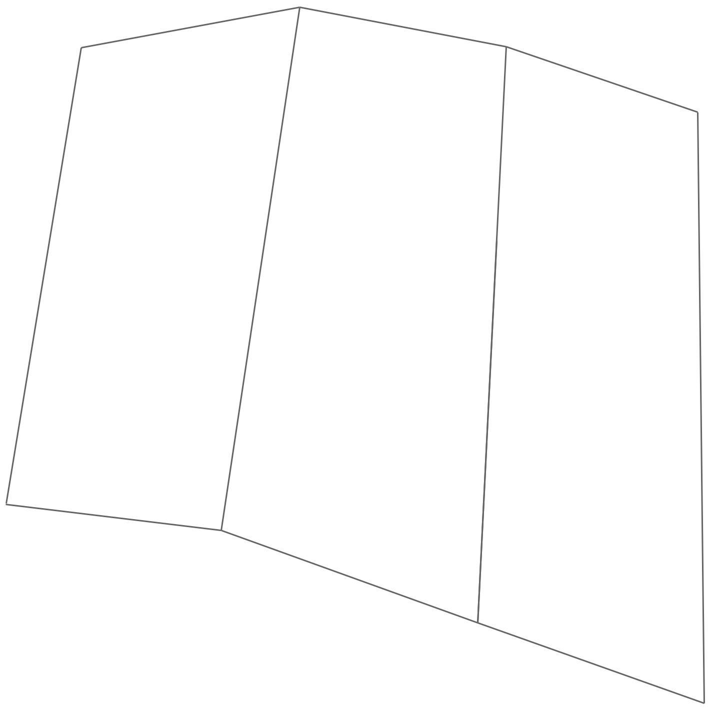
# Discrete Conjugate Ruled



Constant normal along the rulings

Redundant quads and vertices

# Discrete Conjugate Ruled

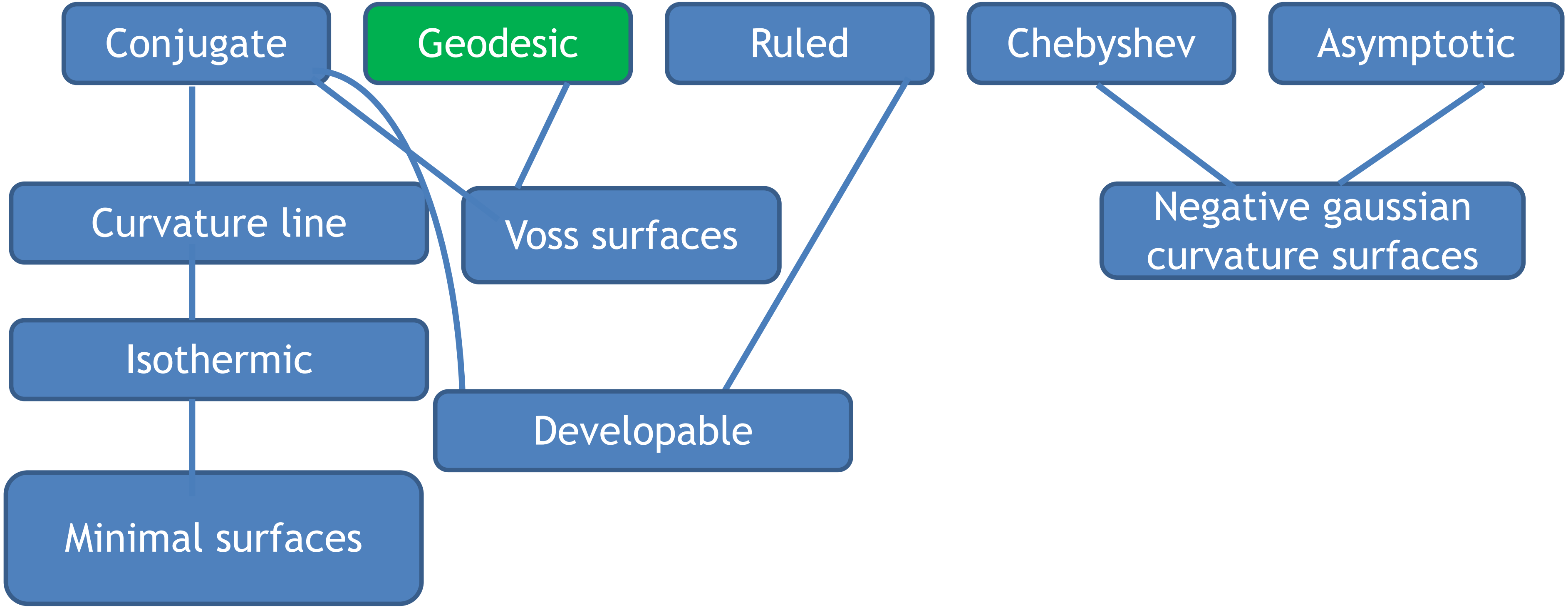


Constant normal along the rulings

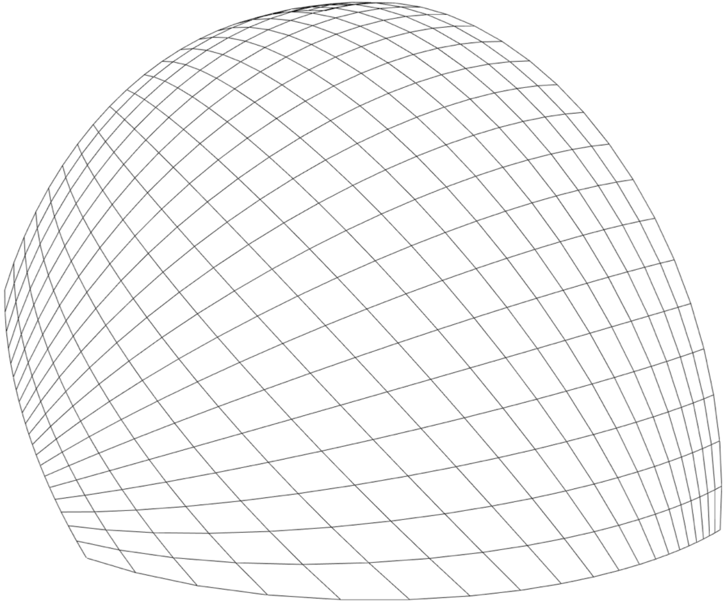
Redundant quads and vertices

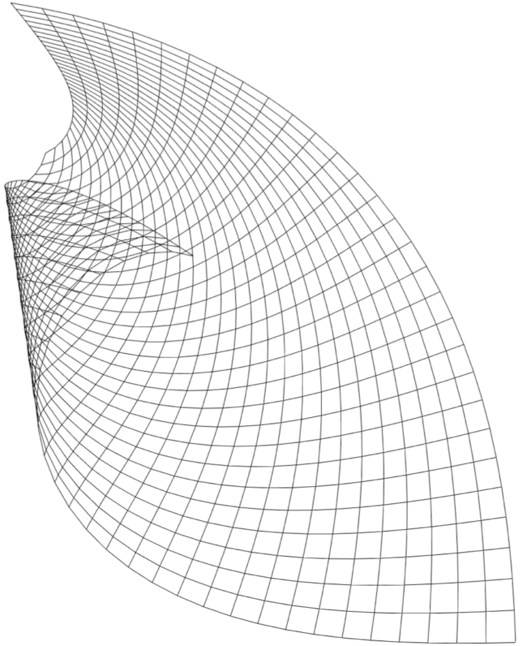
Planar quad strip

# A zoo of discrete nets



# Geodesic nets

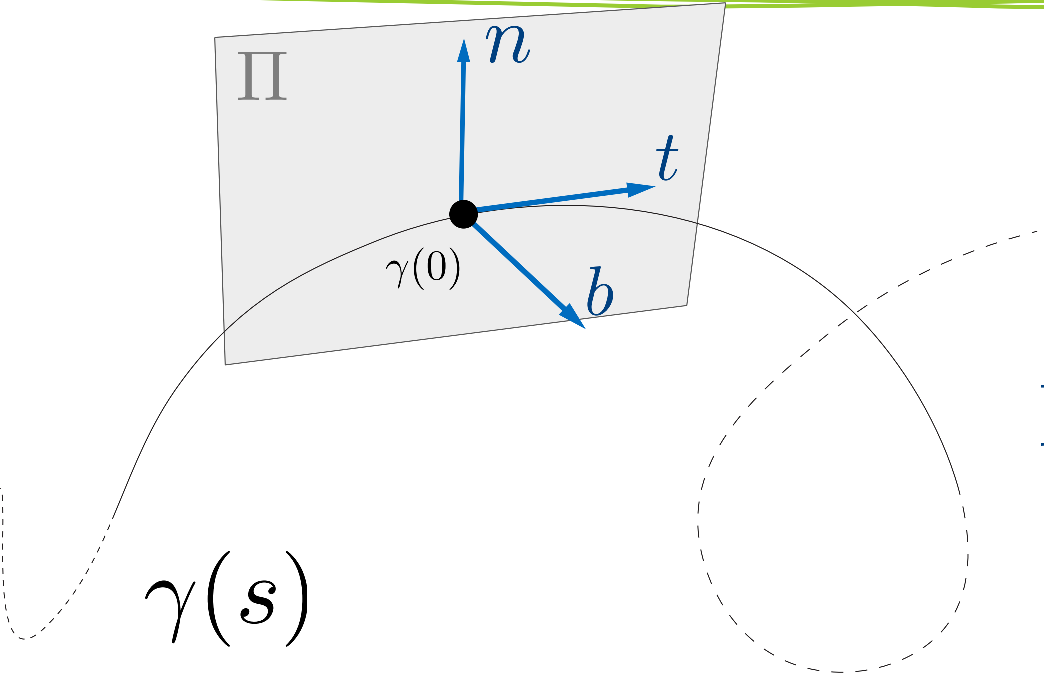




# A geodesic

- Intuitively: As-Straight-As-Possible
- Locally distance minimizer
- Curvature equals normal curvature
- Principal normal is surface normal

# Frenet frame of a curve




$$t = \gamma(0)'$$

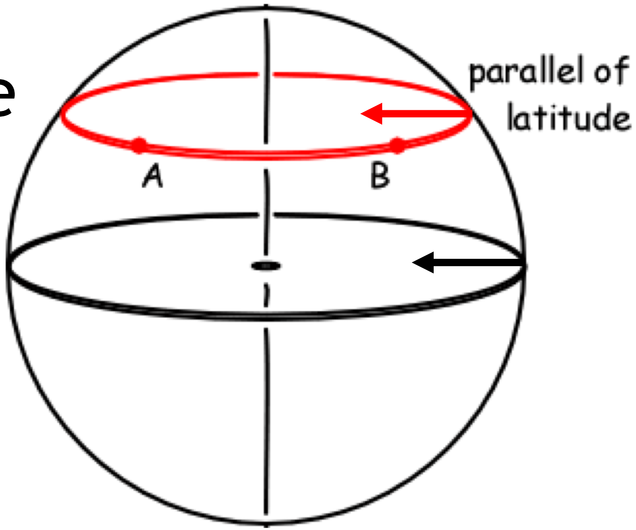
$$n \parallel \gamma(0)''$$

$$\Pi = \gamma(0) + a_1 t + a_2 n$$

$$b \perp t, n$$

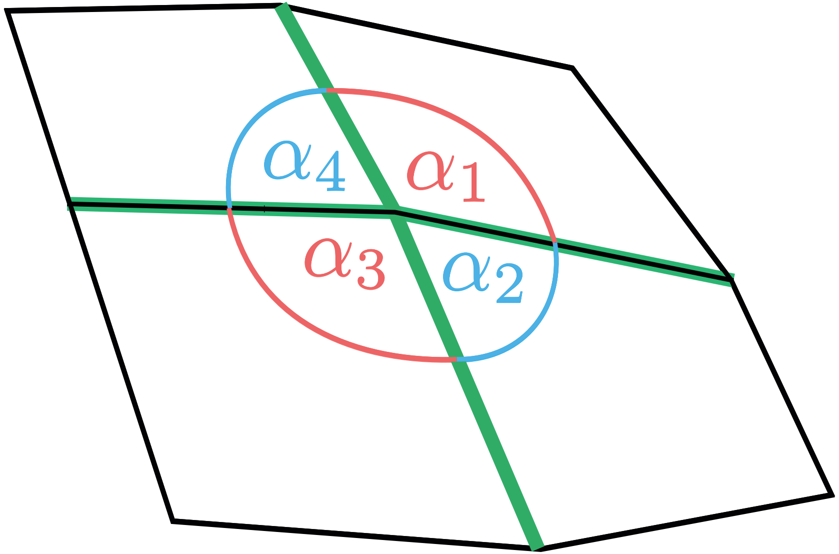
# A geodesic

- Intuitively: As-Straight-As-Possible
- Locally distance minimizer
- Curvature equals normal curvature
- Principal normal is surface normal



# Discrete geodesic nets

- Wunderlich (1951): a geodesic curve is as-straight-as-possible: equal deviation from $\pi$ on both sides




$$\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4$$

$$\alpha_1 + \alpha_4 = \alpha_3 + \alpha_2$$

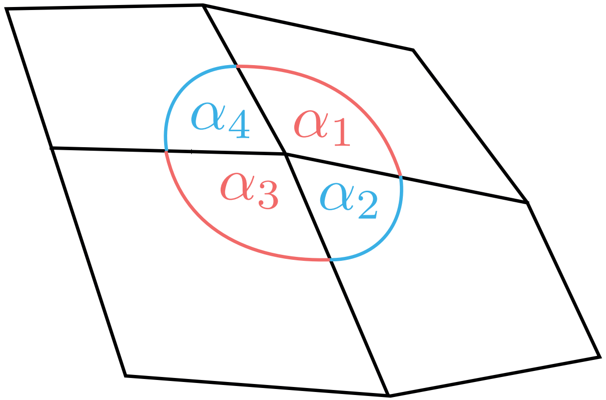
$$\Downarrow$$

$$\alpha_1 = \alpha_3$$

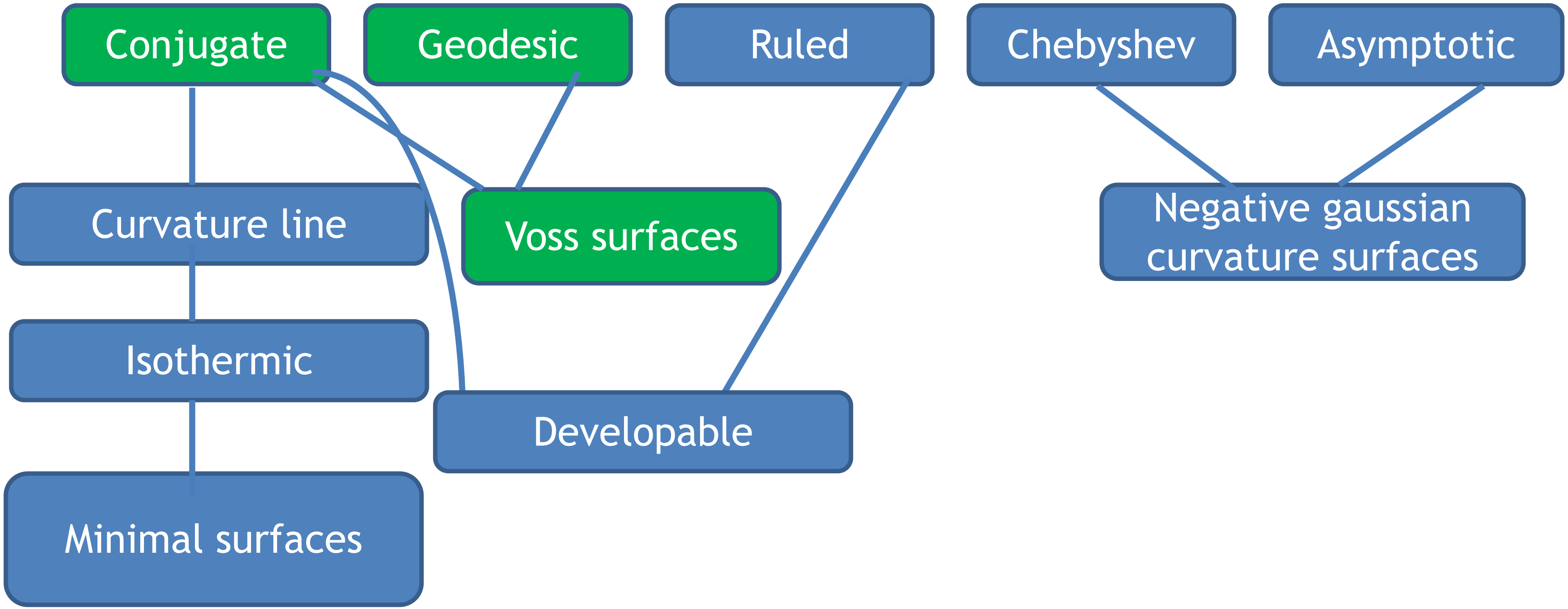
$$\alpha_2 = \alpha_4$$

# Discrete geodesic nets

- Discrete curvature normal is surface normal
- Not necessarily planar (not conjugate)



# A zoo of discrete nets



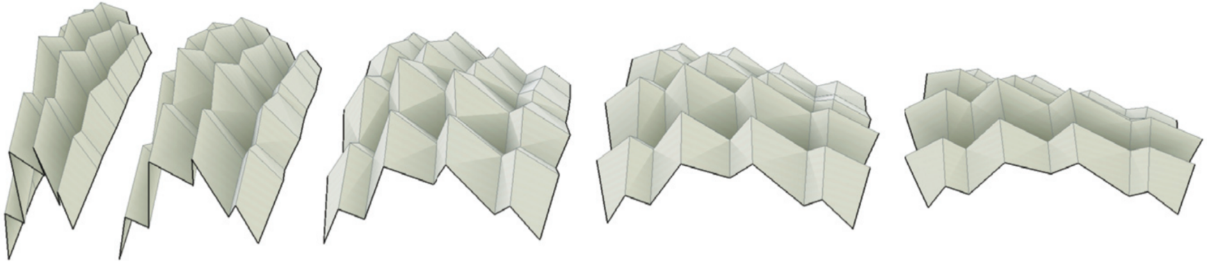
# Voss Surfaces

- Geodesic + Conjugate
  - Can be (uniquely) isometrically deformed

[Wunderlich 1951]
[Wolfgang K. Schief et al. 2008]]

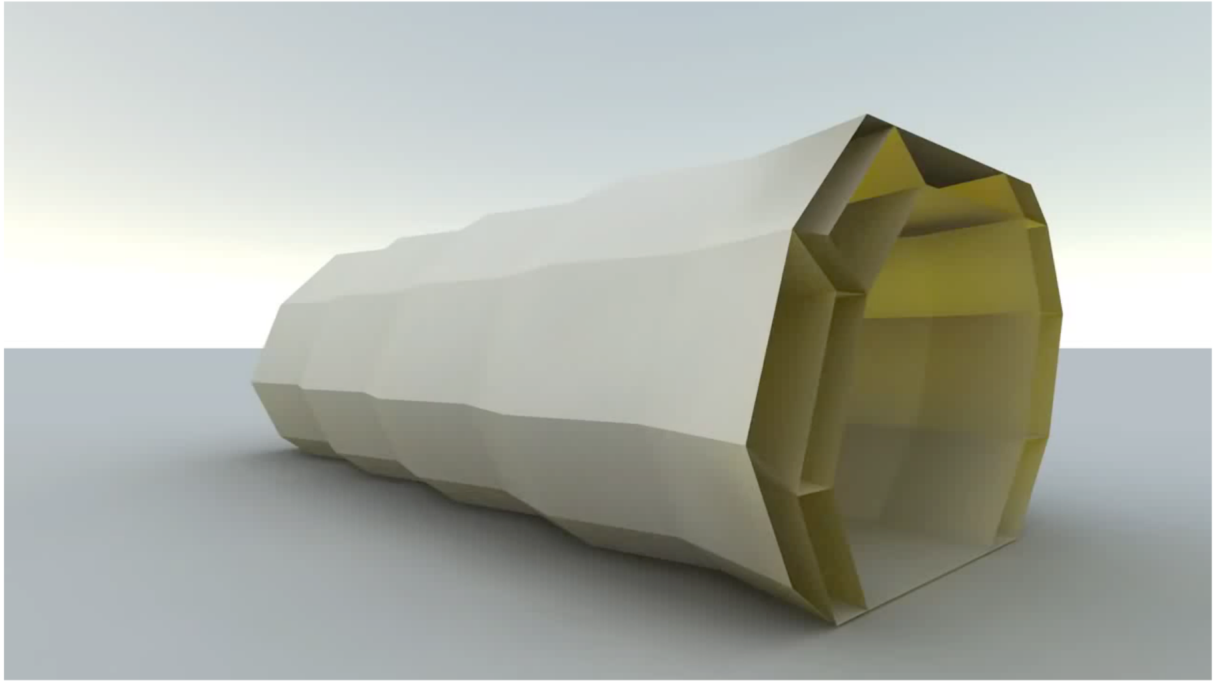
# Discrete Voss Surfaces

- Planar and opposite angles equal
  - Can be isometrically deformed [Schief et al. 2007]
  - Origami and deployable structures



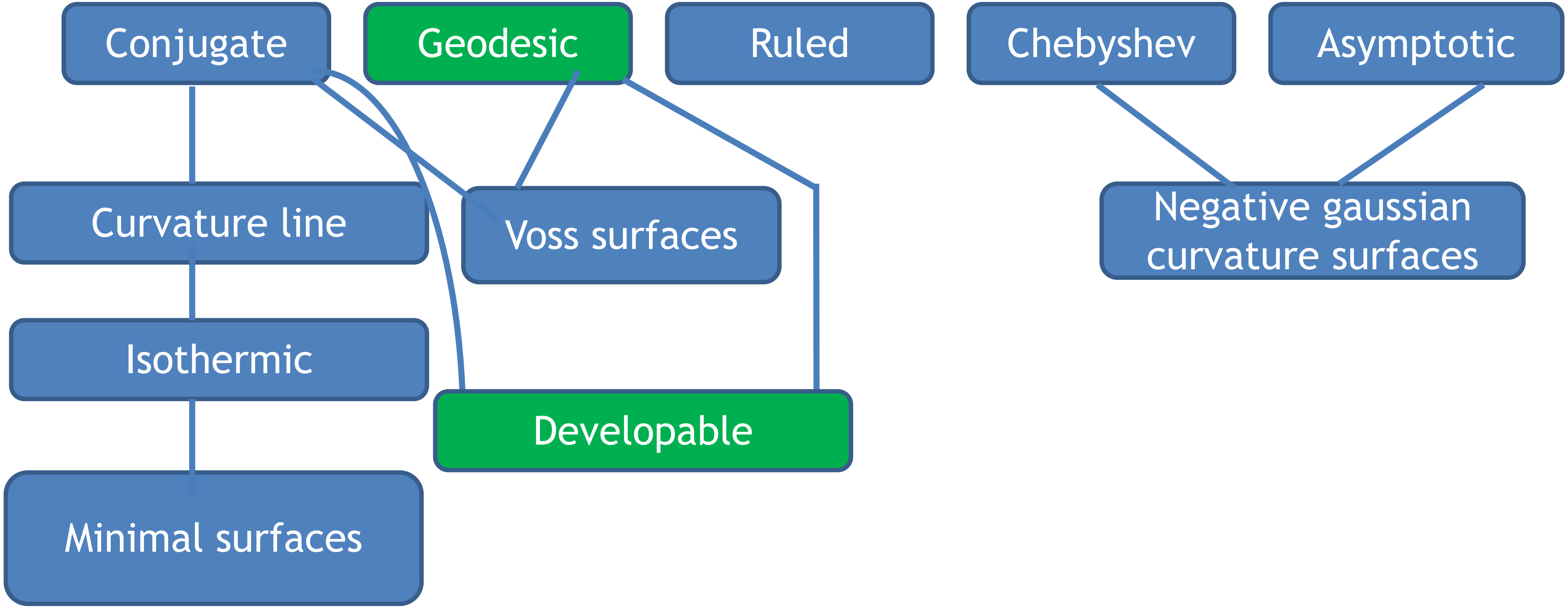
[Freeform Rigid-Foldable Structure using Bidirectionally Flat-Foldable Planar Quadrilateral Mesh, Tomohiro Tachi 2010]

# Rigid Fold Tube Origami



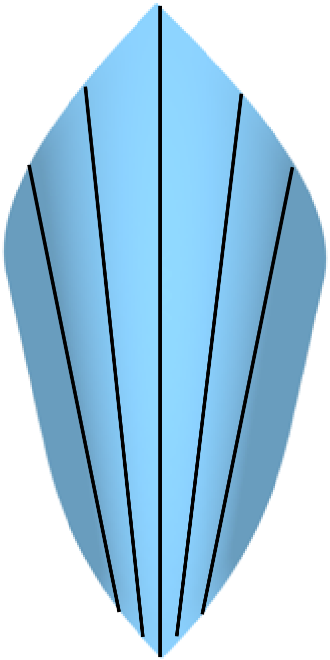
Tomohiro Tachi

# A zoo of discrete nets

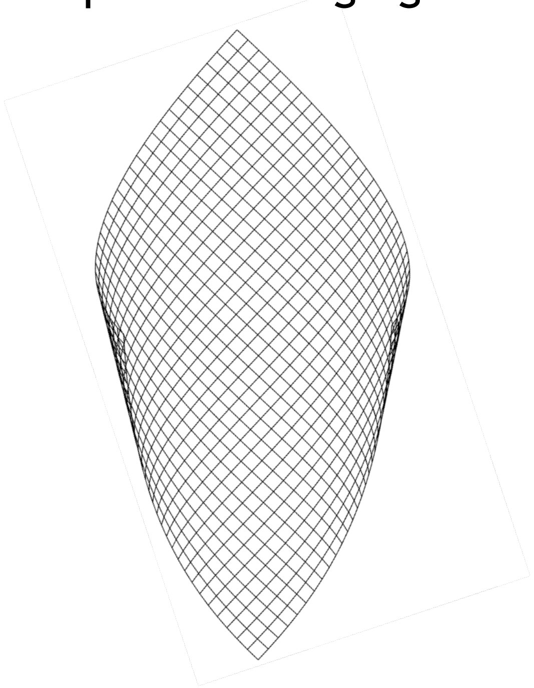


# Developable surface through geodesics

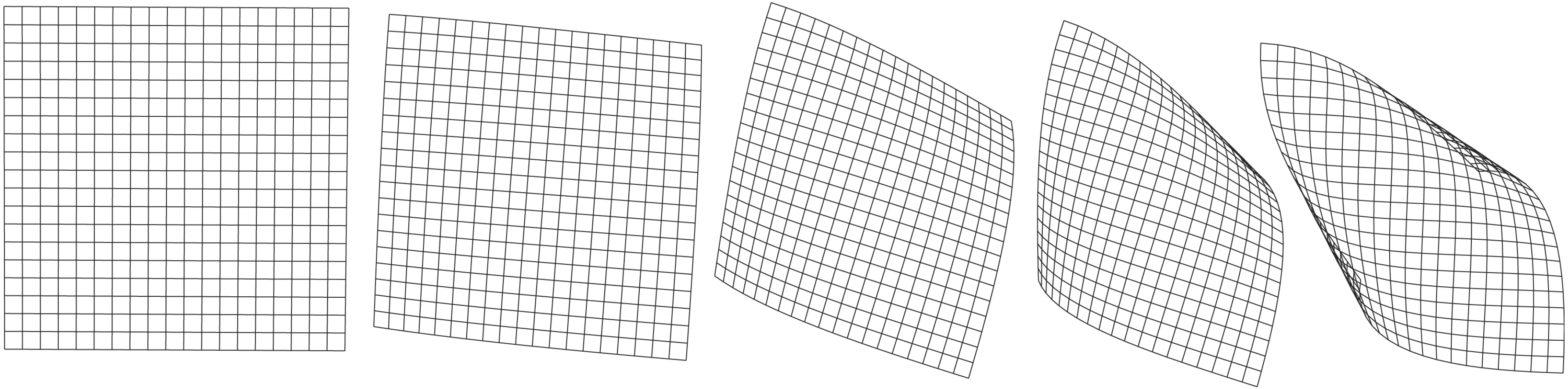
Developable through rulings



Developable through geodesics



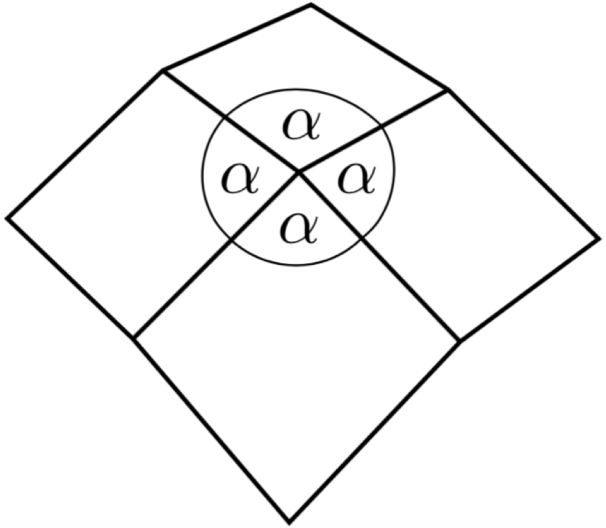
# Developable surface through orthogonal geodesics

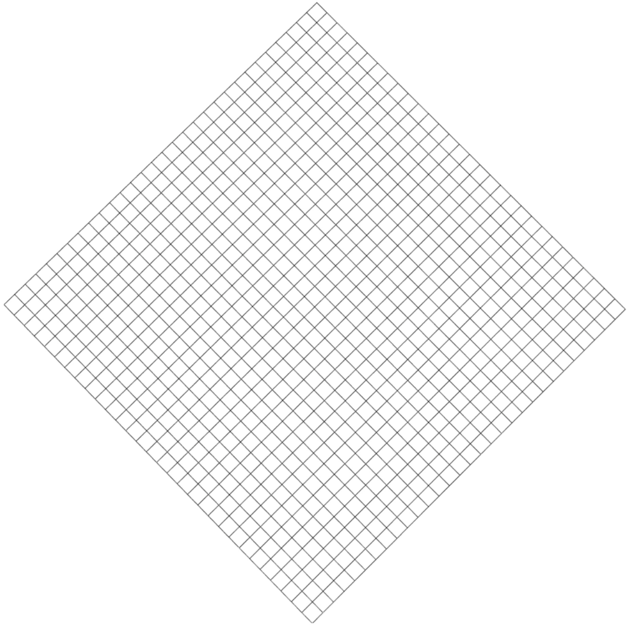


Developable ⟺ Admits orthogonal geodesic net

[Rabinovich et al. 2018]

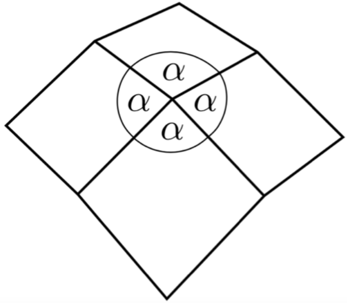
# Discrete orthogonal geodesic net



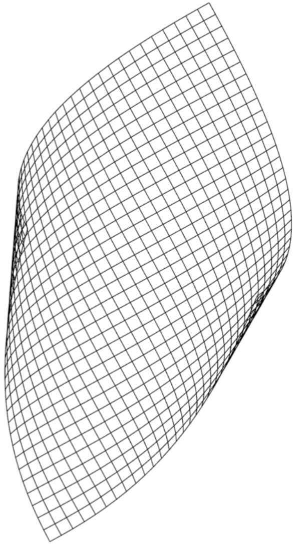



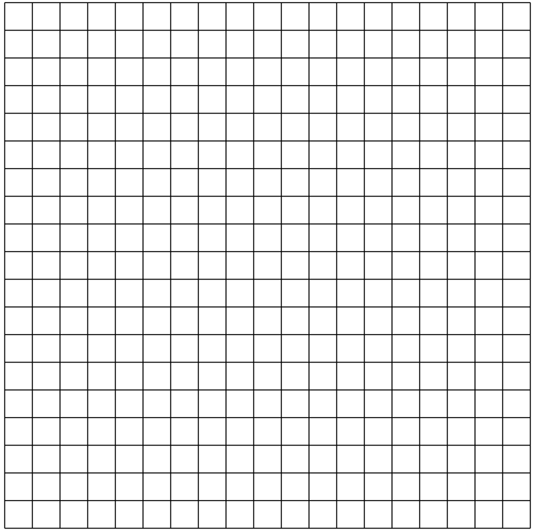
# Avoids Locking

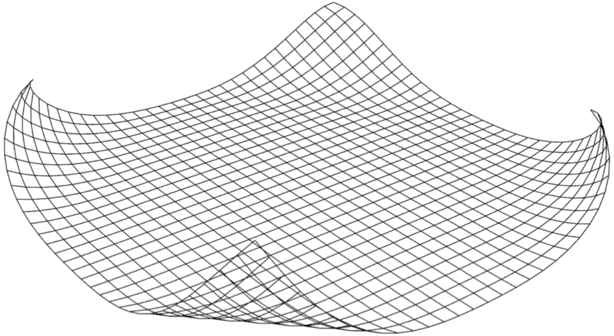
Fixed intrinsic grid meshing with a set of angle constraints



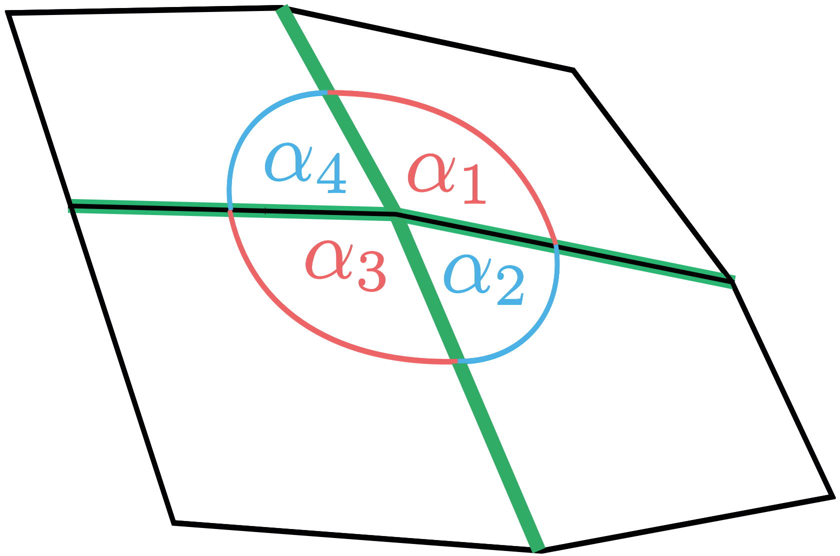

[Rabinovich et al. 2018]



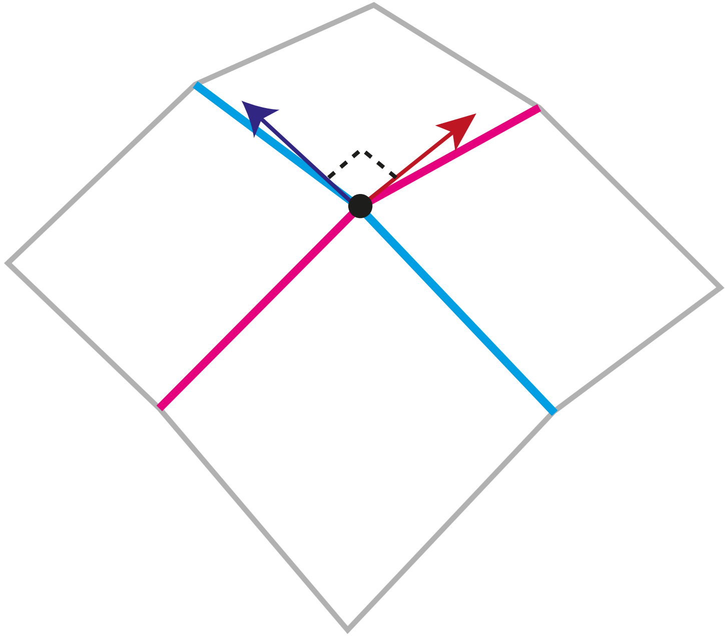




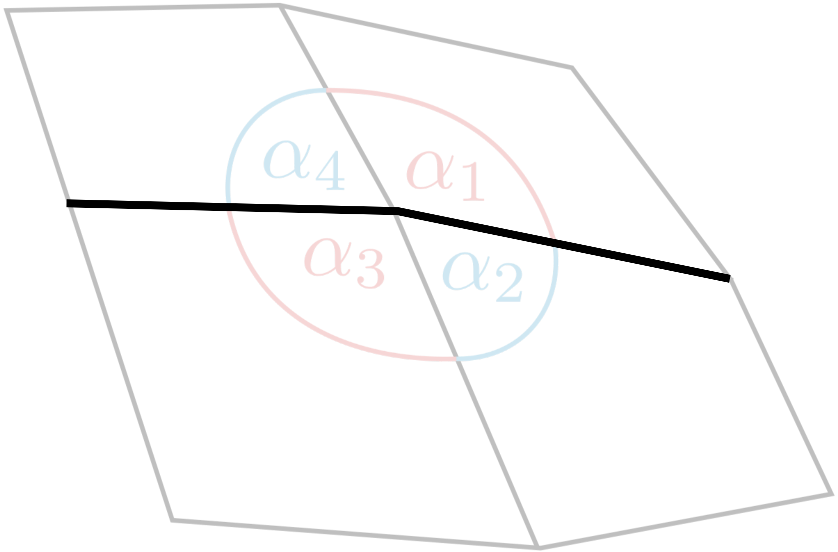
# Discrete geodesic nets



# Geodesics meet orthogonally

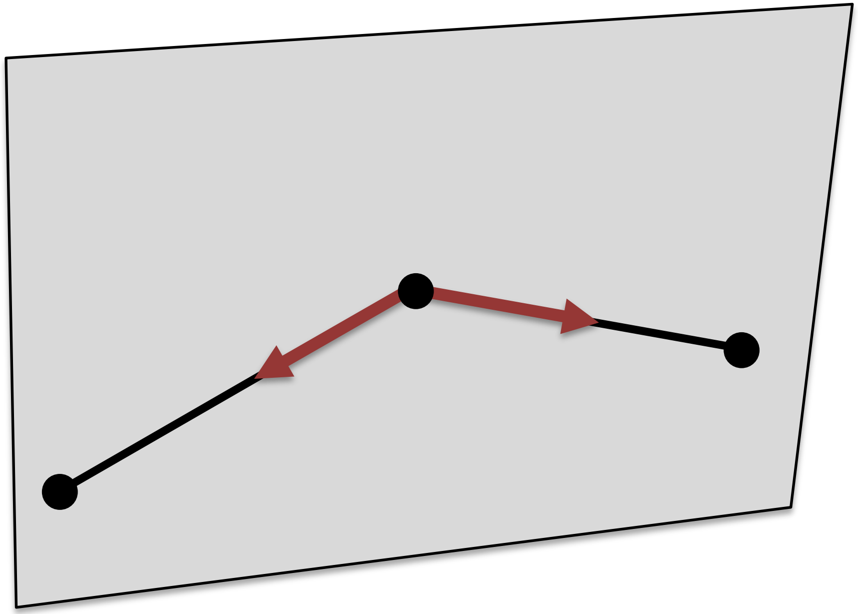


# Discrete geodesic nets

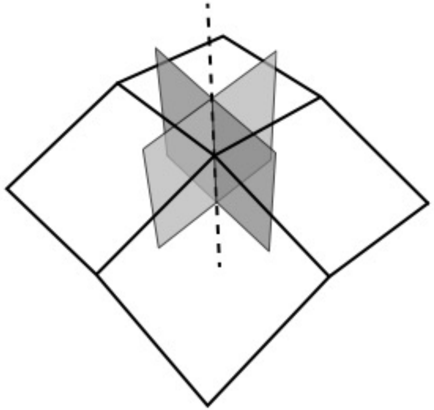


# Osculating planes of geodesics

Osculating plane of the coordinate curve



**Intersecting geodesics**



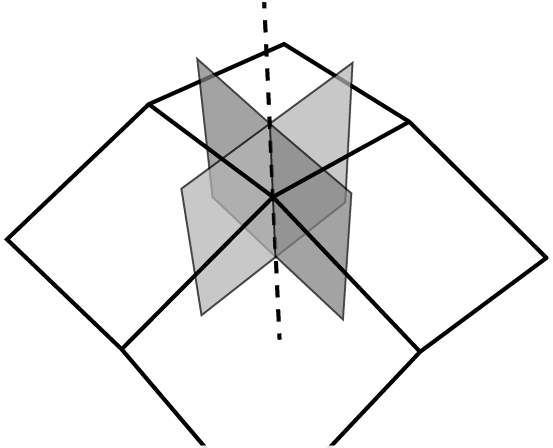
orthogonal tangents

⇕

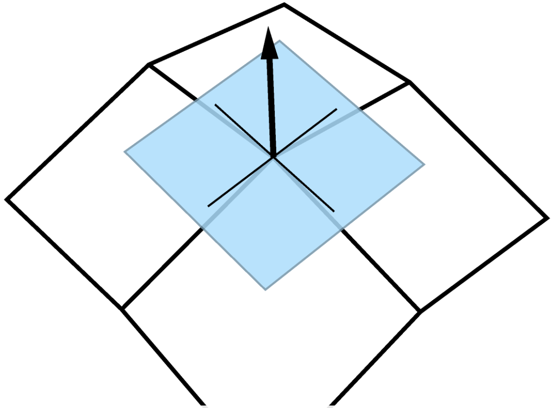
orthogonal osculating planes

# Discrete *orthogonal* geodesic nets

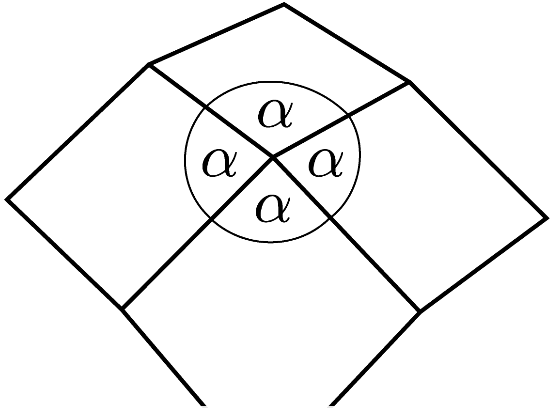
- Given a discrete geodesic net (opposite angles equal), these 3 conditions are equivalent:



osculating planes intersect orthogonally

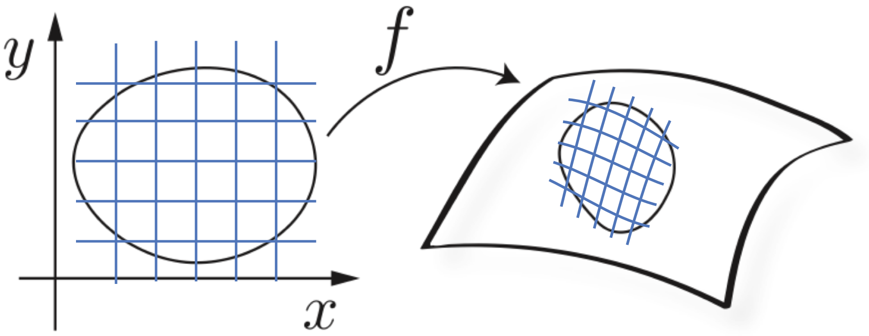


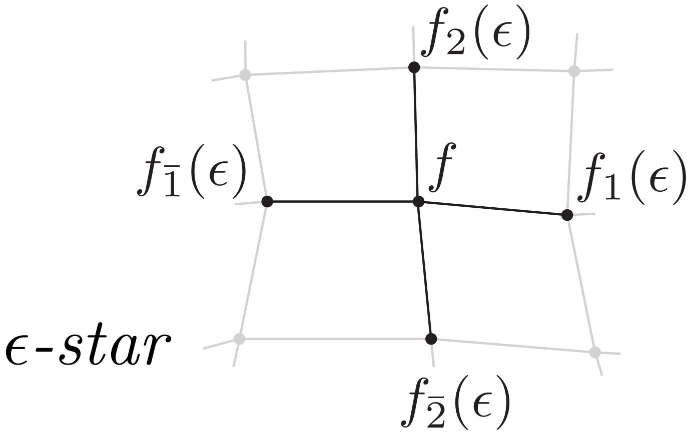
edges projected onto tangent plane form orthogonal cross

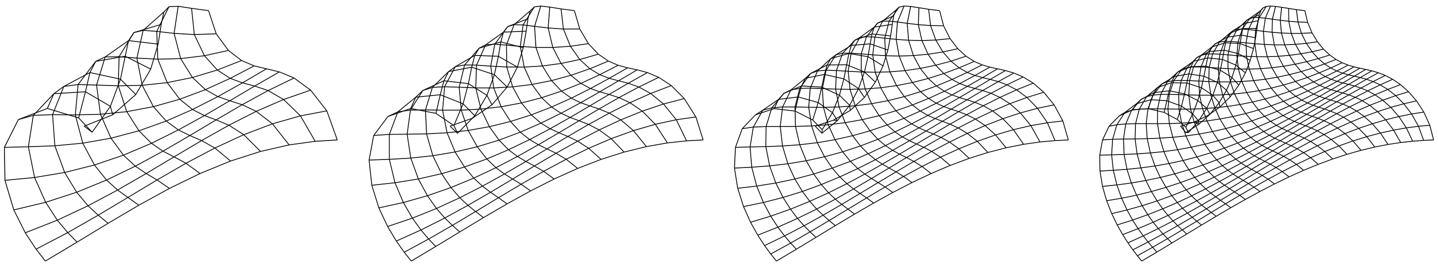



all angles around the star are equal

# Infinitesimal stars of a smooth net



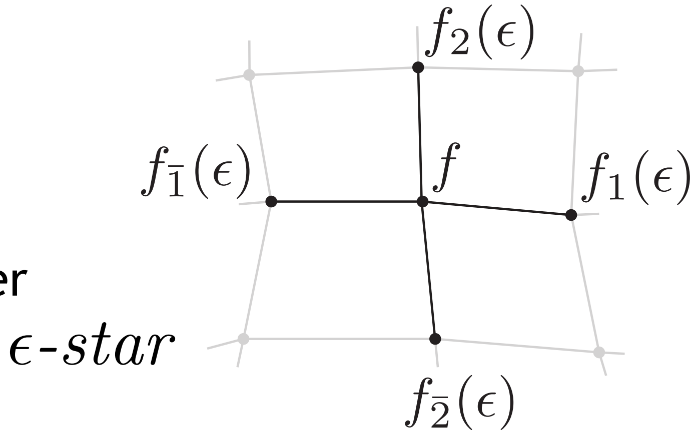



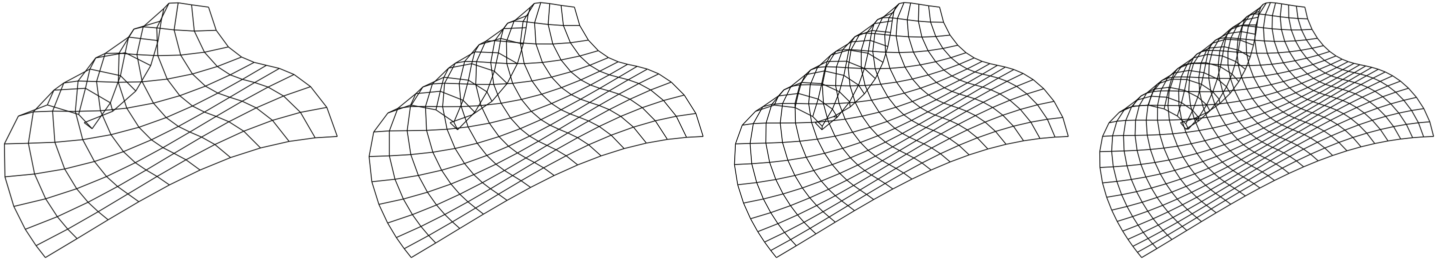
# Infinitesimal stars of a smooth net

A smooth net $f$ is an orthogonal geodesic net

⇕

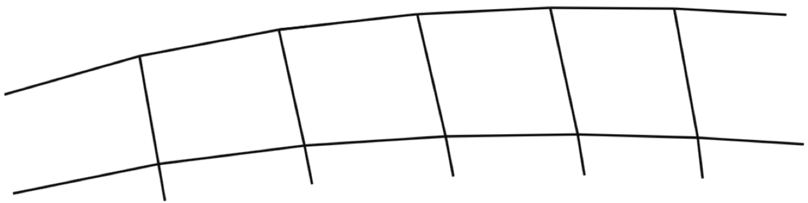
Its $\epsilon$-$stars$ have equal angles up to second order



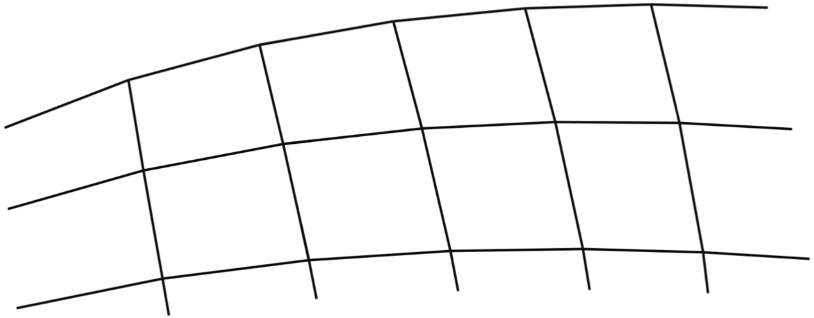



# Discrete developable evolution

Input

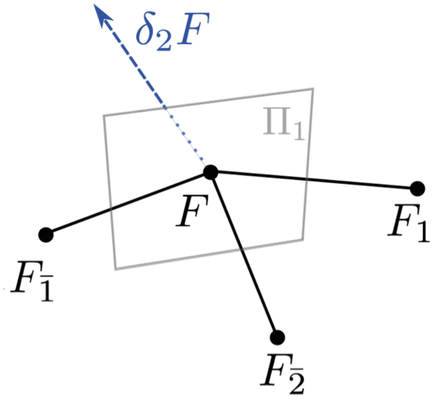


Output

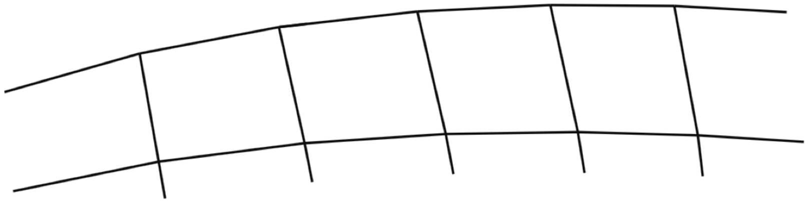


# Discrete developable evolution

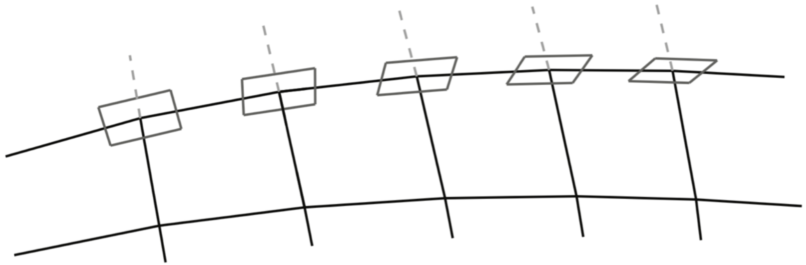
Direction propagation lemma



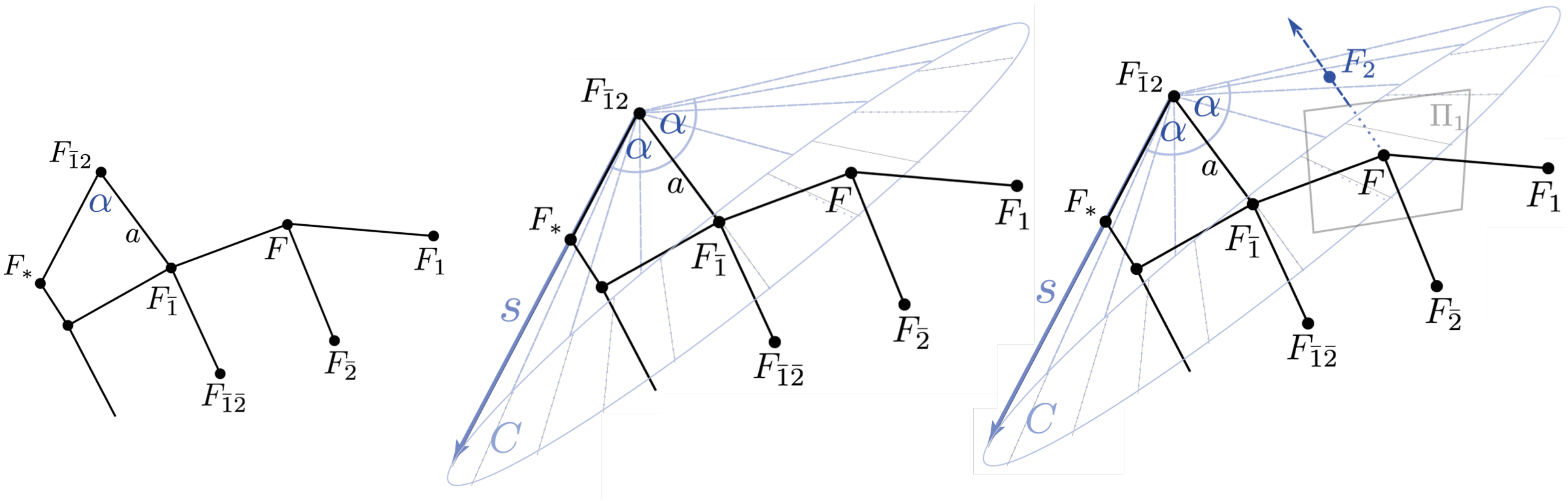

Input



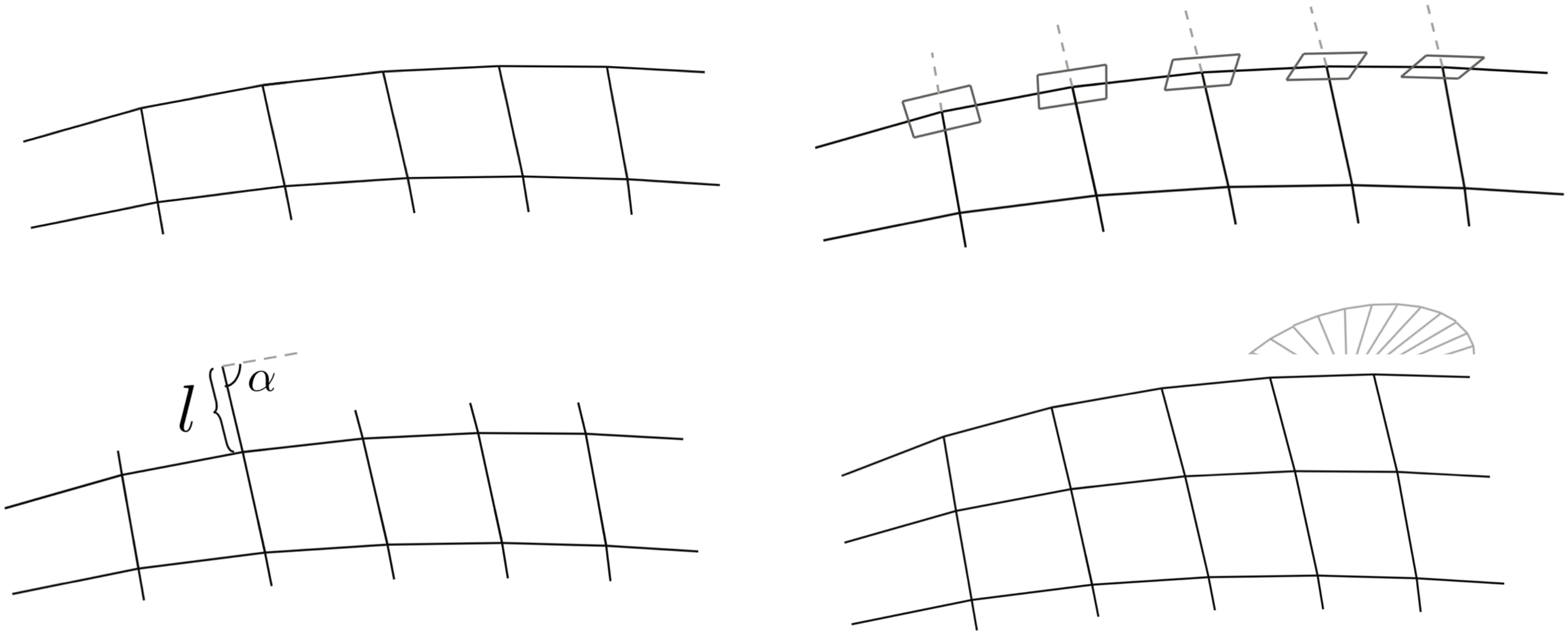
Direction propagation

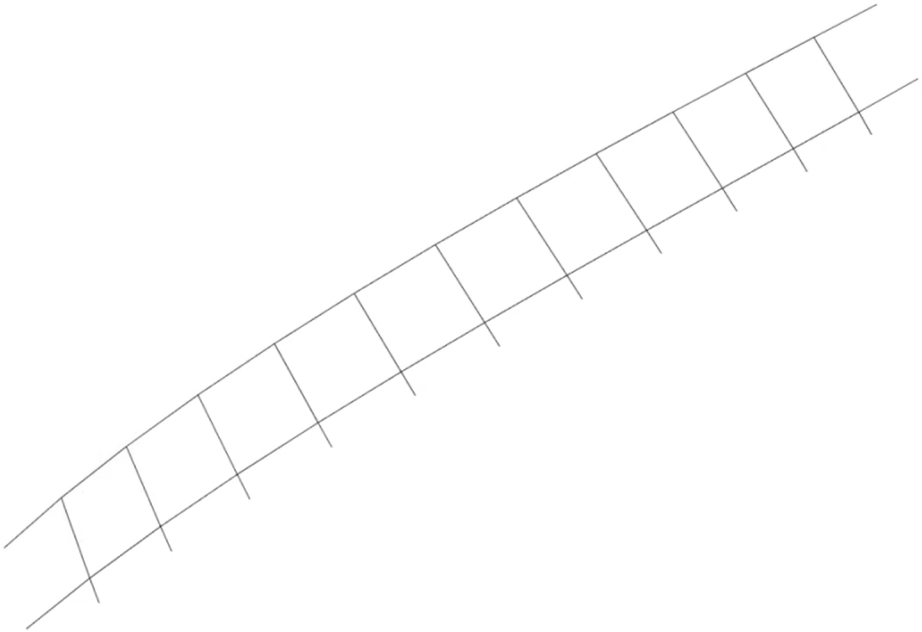


# Cone intersections

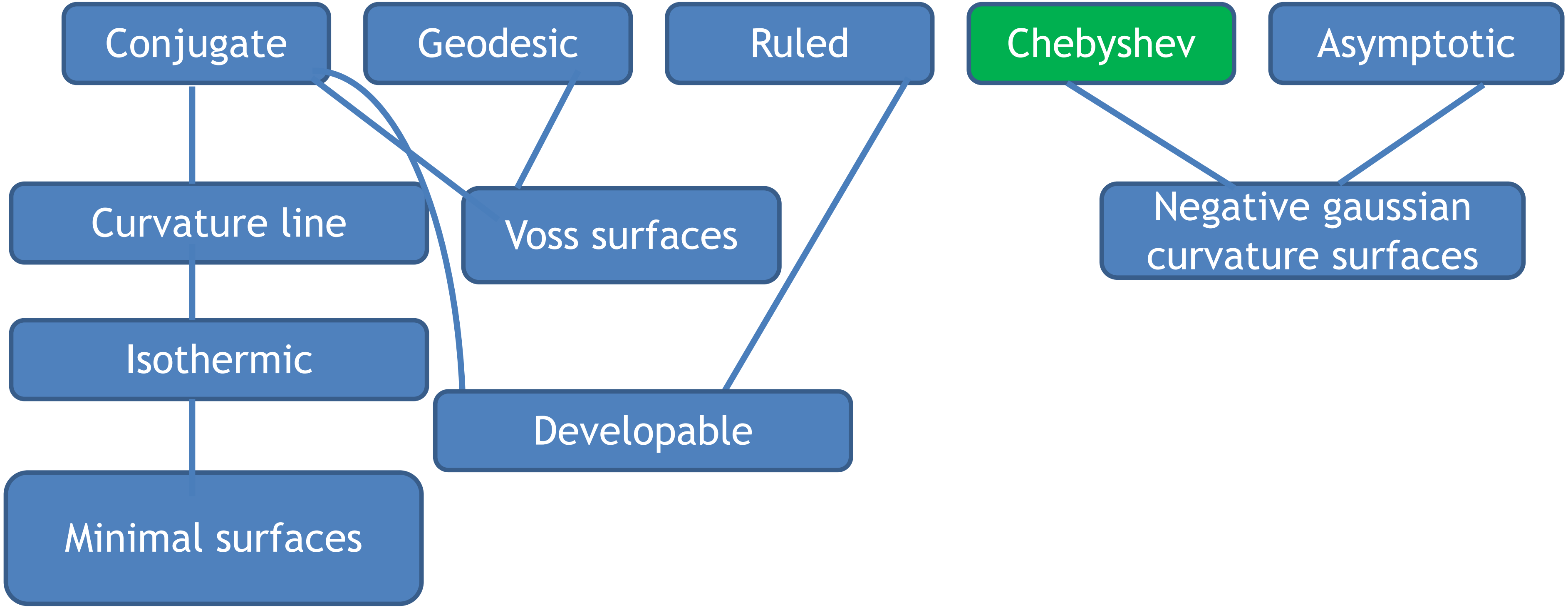


# Discrete developable surface extension





# A zoo of discrete nets



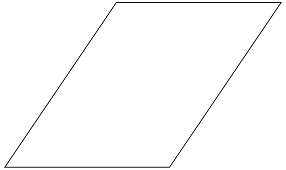
# Chebyshev nets

## Smooth

$$\|f_x\|_y = \|f_y\|_x = 0$$

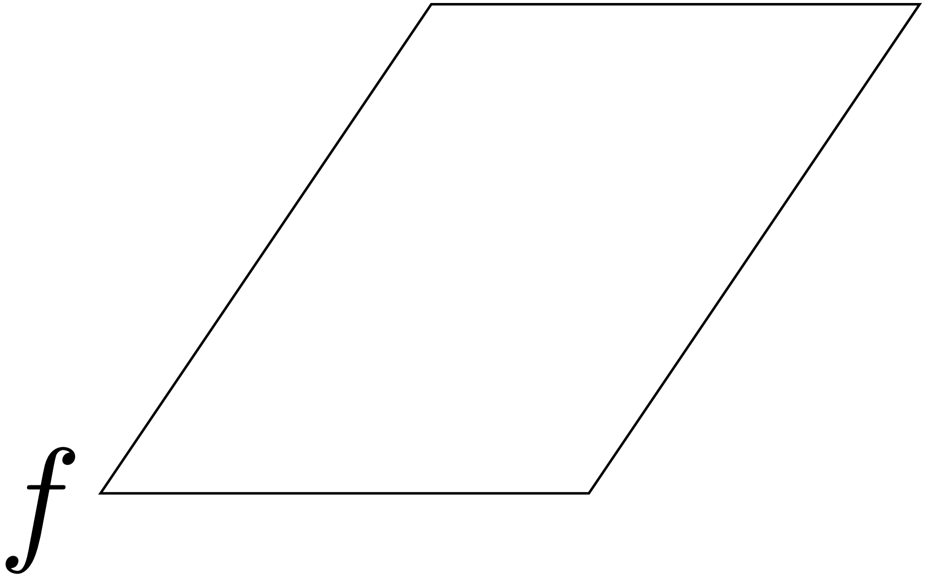
## Discrete

Parallelogram quad net

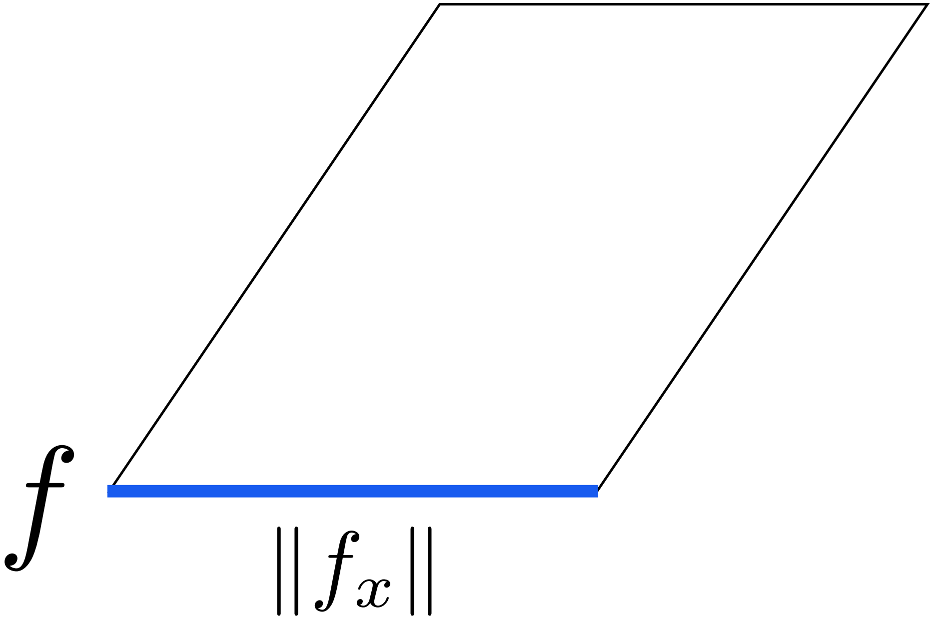


[Akash Garg, et al. 2014]

$$\|f_x\|_y = \|f_y\|_x = 0$$

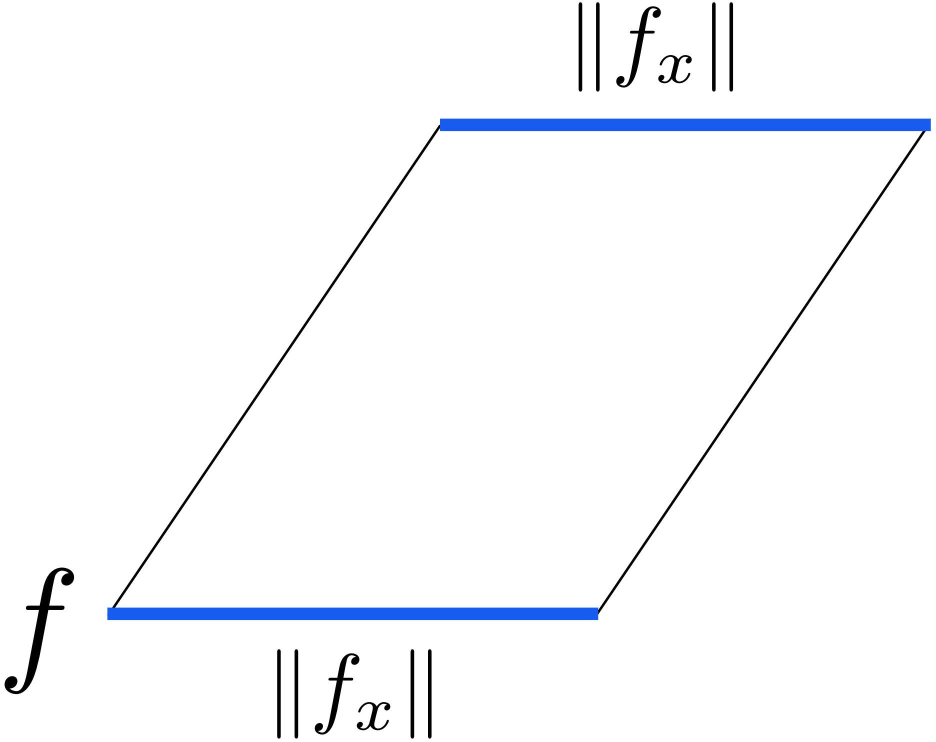


$$\|f_x\|_y = \|f_y\|_x = 0$$



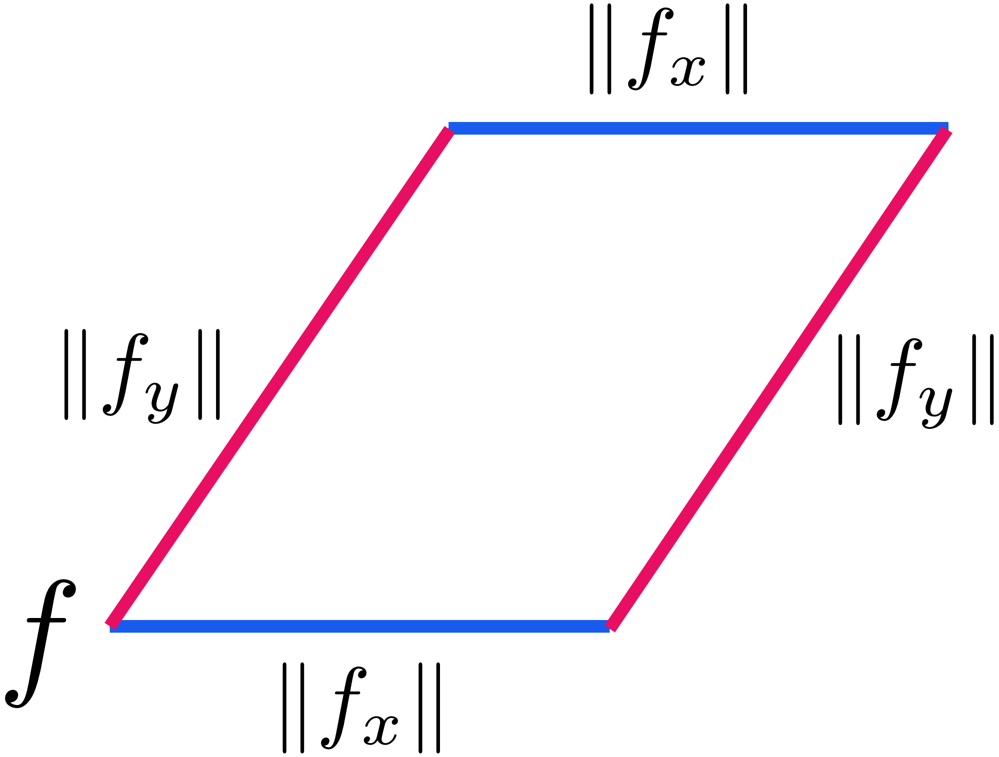

$$\|f_x\|_y = 0$$

$$\|f_x\|_y = \|f_y\|_x = 0$$




$$\|f_x\|_y = 0$$

$$||f_x||_y = ||f_y||_x = 0$$

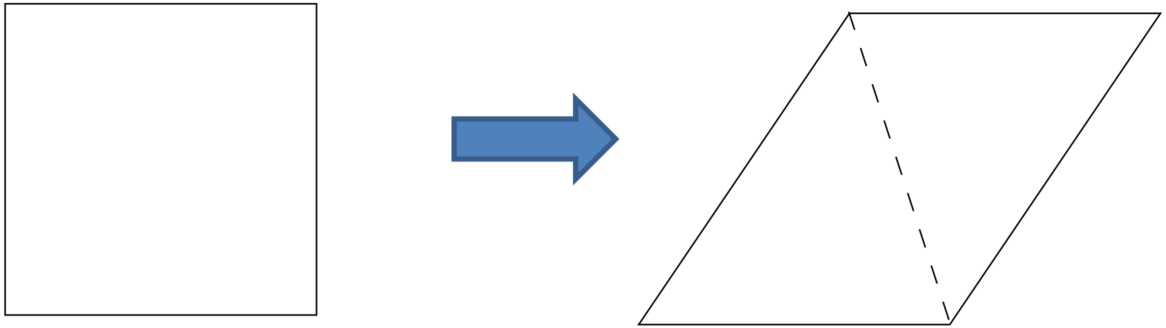



$$||f_x||_y = 0$$

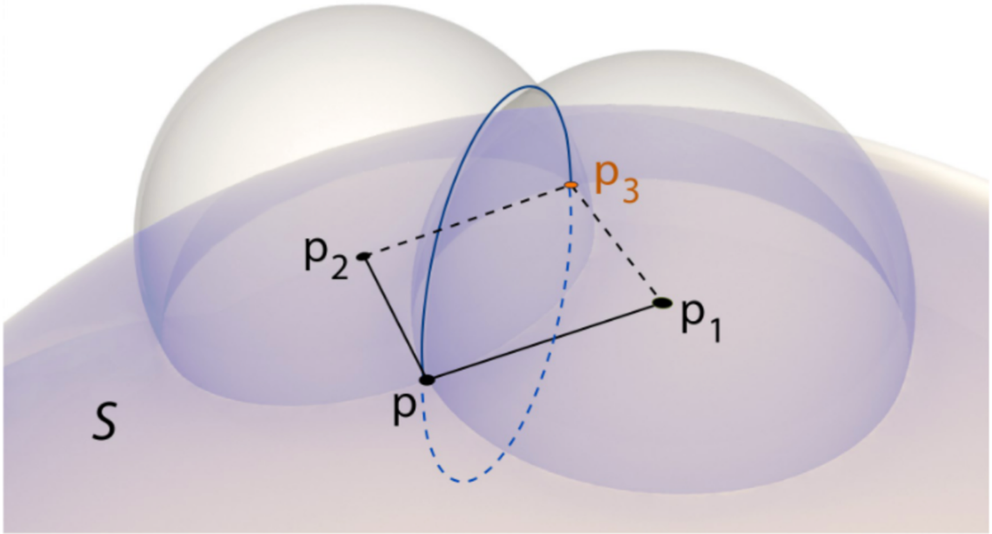
$$||f_y||_x = 0$$

# Chebyshev net

- Any surface can be locally parameterized
  - Curvature through shear and bending
  - But not globally



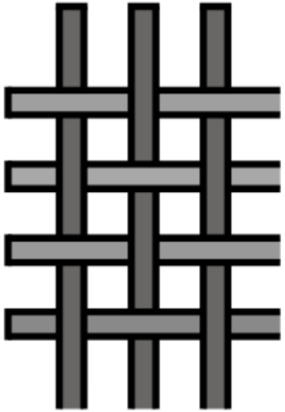
# Chebyshev construction

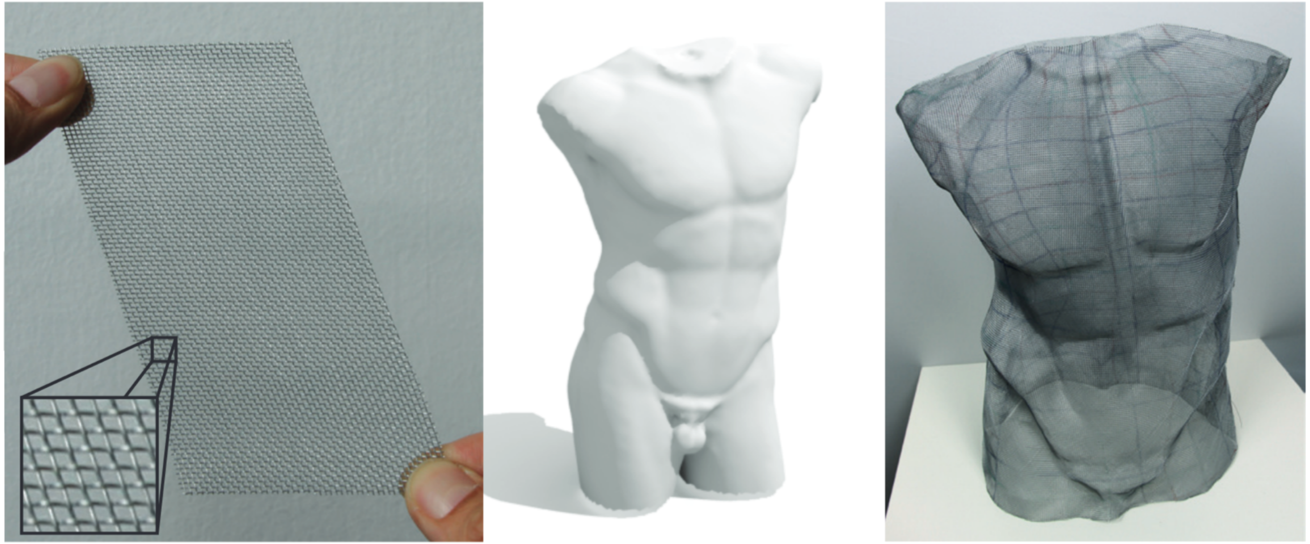



[Wire Mesh Design, Akash Garg, et al. 2014]

# Chebyshev net applications

Metal wires



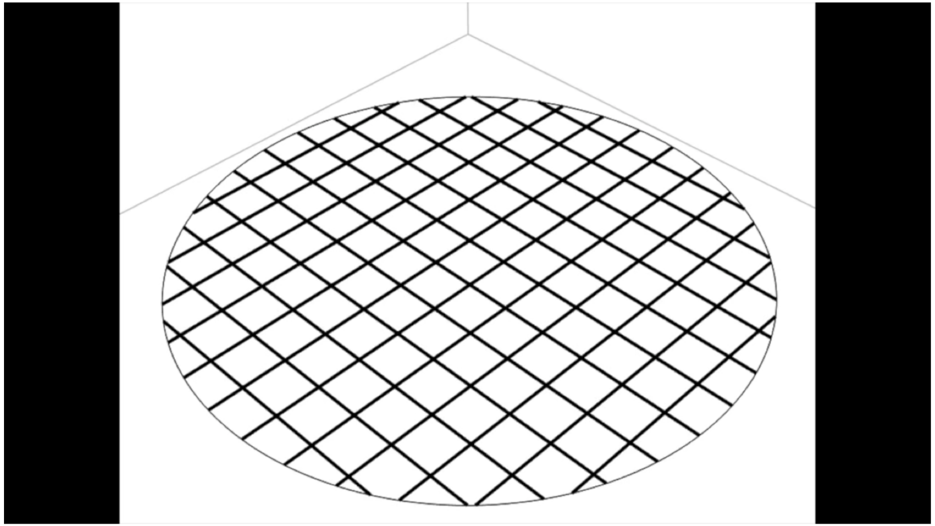


[Wire Mesh Design, Akash Garg, et al. 2014]

# Chebyshev net applications

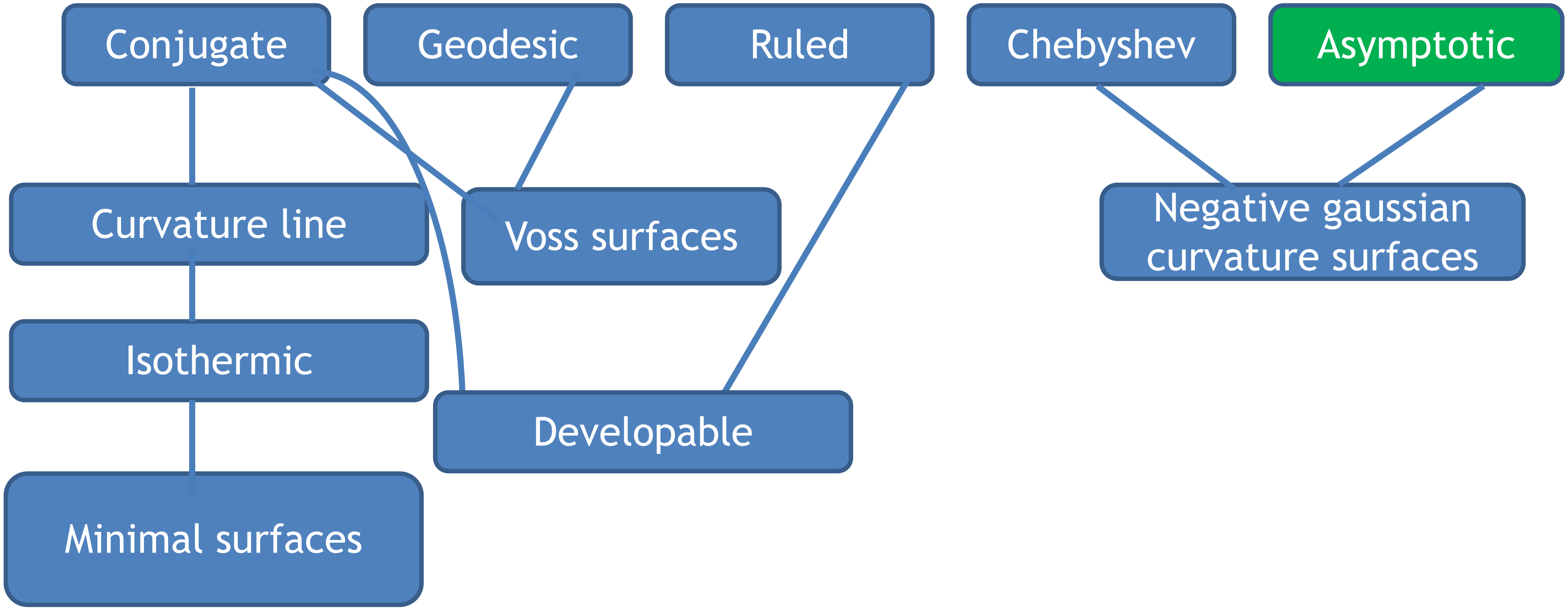
Elastic grid shells

[Baverel, Caron, Tayeb, Du Peloux, 2012]



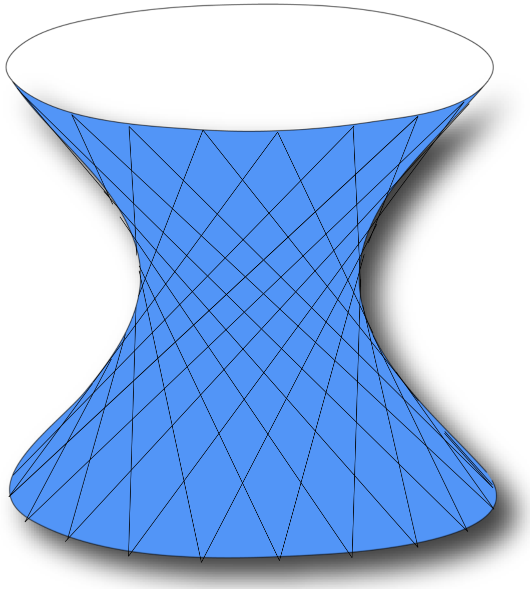
[Form finding in elastic gridshells, Changyeob Baek et al. 2018]

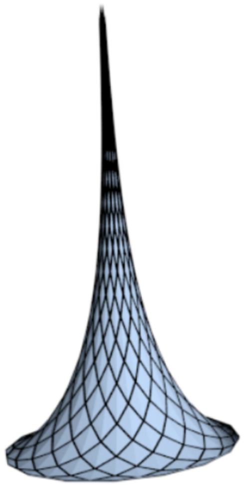
# A zoo of discrete nets



# Asymptotic nets

Parameterized by curves with 0 normal curvature





[Hoffmann et al. 2014]

# Asymptotic nets

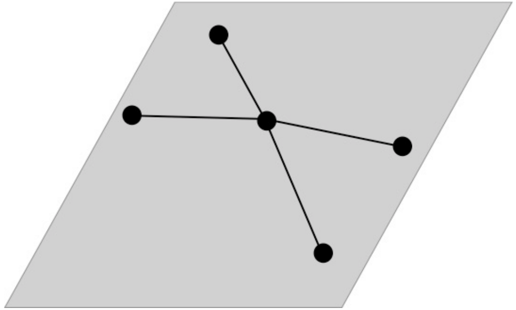
**Smooth**

$f_x, f_y$ Directions of 0 normal curvature

$$\langle f_x, n_x \rangle = \langle f_y, n_y \rangle = 0$$

**Discrete**

Planar stars



Affine invariance

[Alexander I.Bobenko and Ulrich Pinkall 1999]

# Planar stars

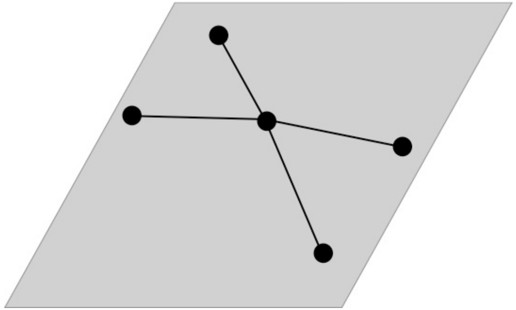
$$\langle f_x, n_x \rangle = \langle f_y, n_y \rangle = 0$$

⇕

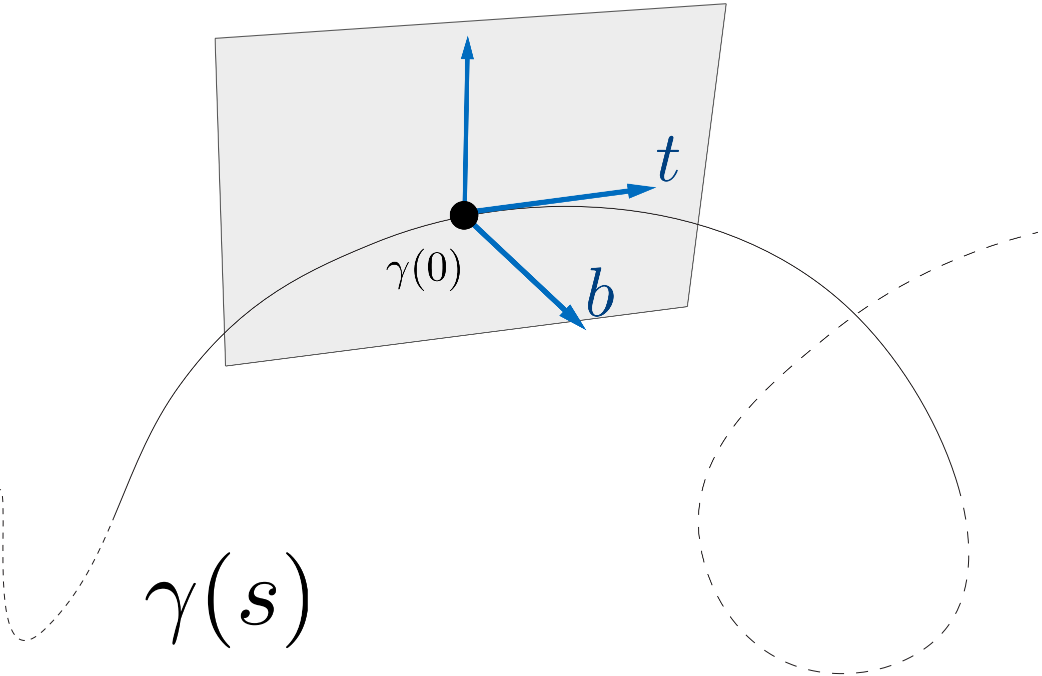
$$f_{xx} \perp n, f_{yy} \perp n$$

⇕

Stars are planar up to second order



# Osculating Planes are Tangent Planes



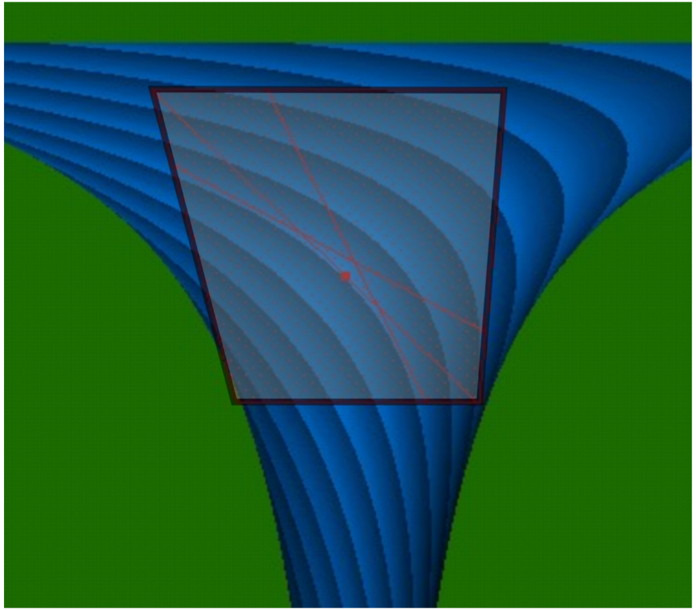
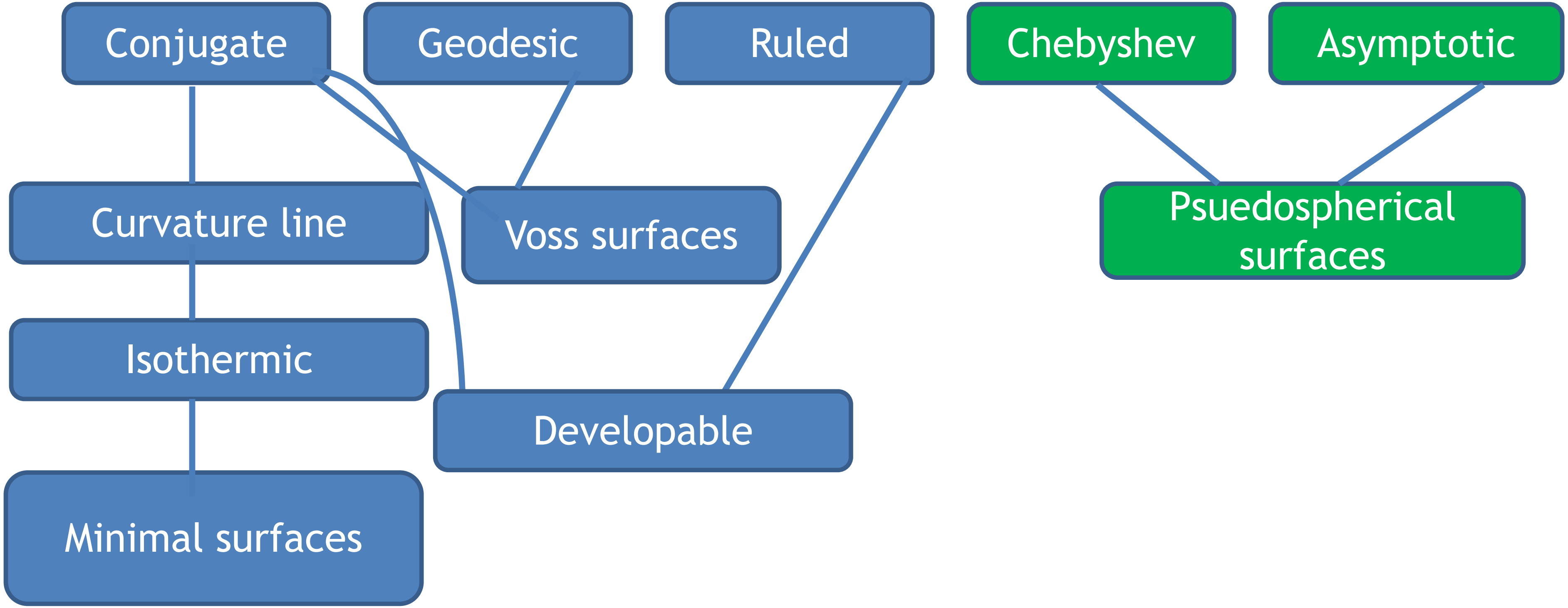



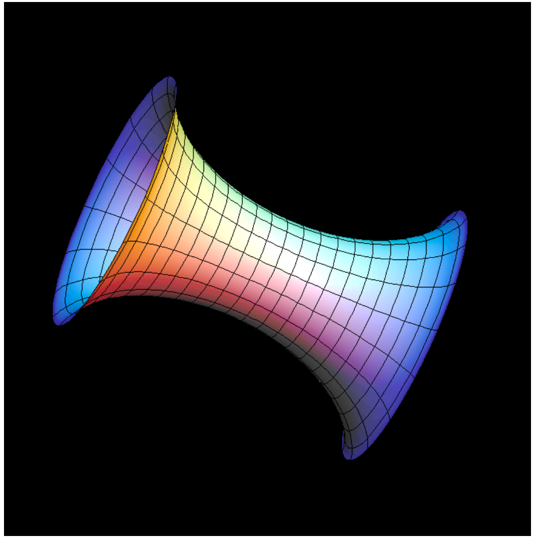
Image taken from [Asymptotic Path Curves N.C. Thomas]

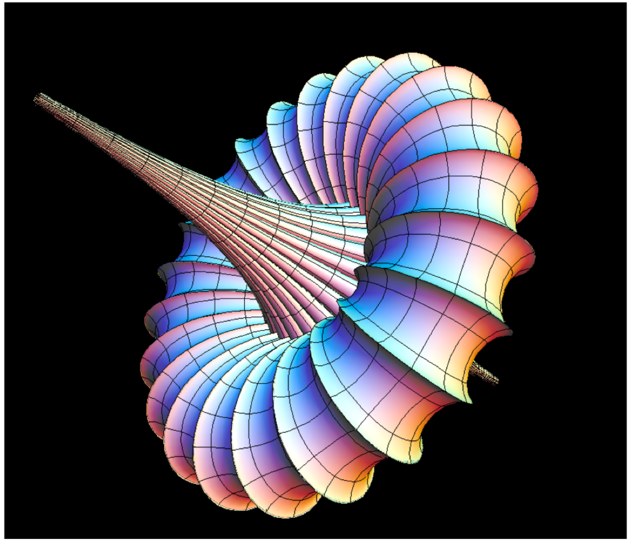
# A zoo of discrete nets

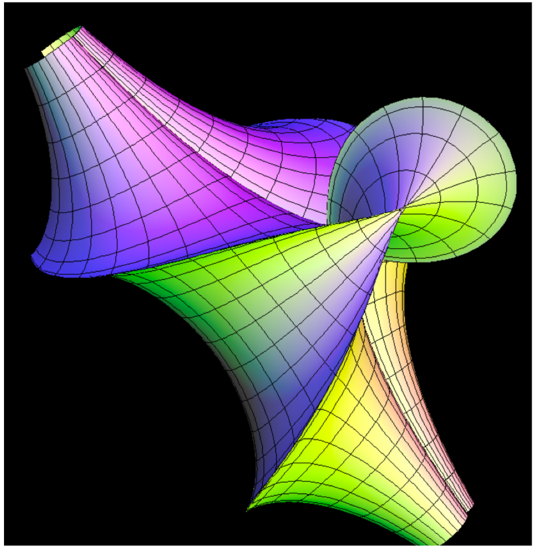


# Chebyshev + Asymptotic = Pseudospherical

$$K = -1$$



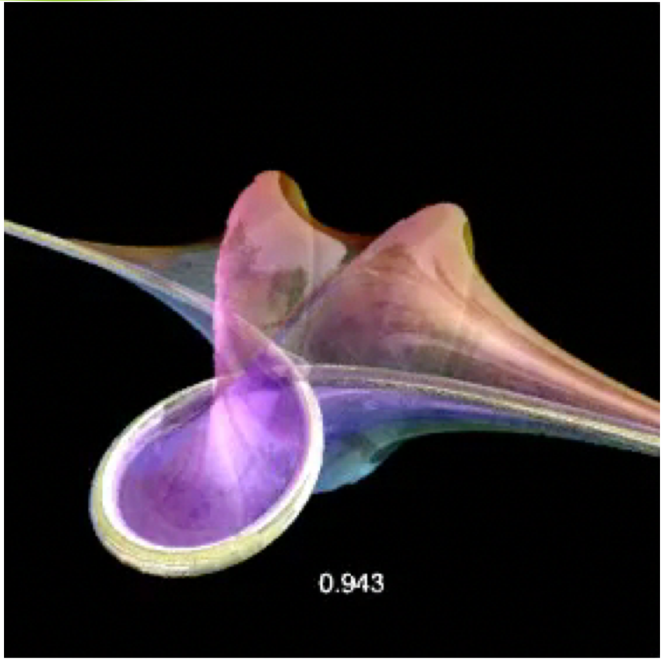




Virtual Math Museum
http://virtualmathmuseum.org/index.html

# (Discrete) Psuedospherical Surfaces

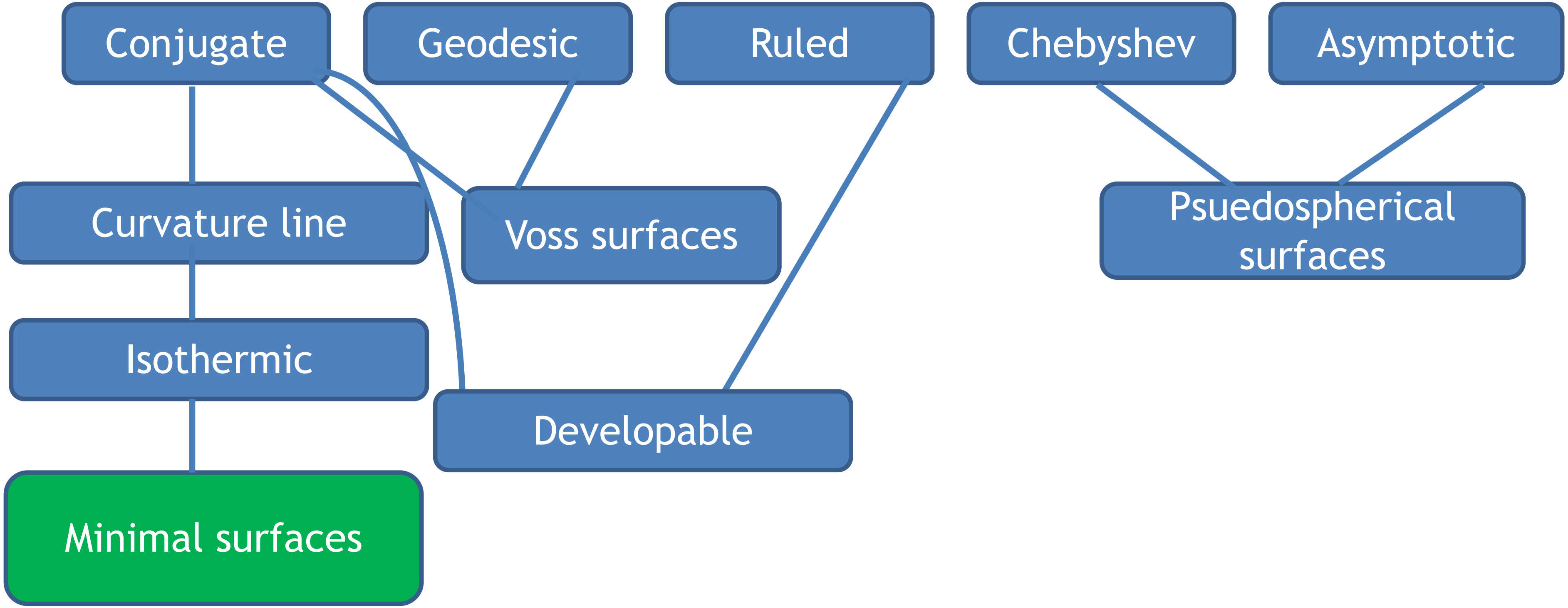
- Deformable
- Dual to Voss Surfaces
- Soliton theory
  - Integrable systems




Virtual Math Museum
http://virtualmathmuseum.org/index.html

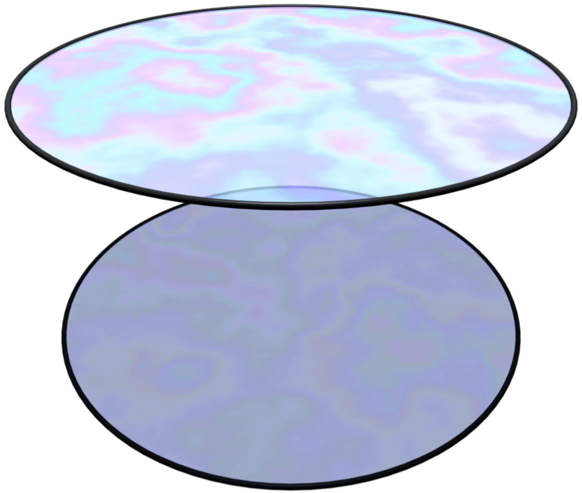
[Wunderlich 1951]
[Alexander Bobenko and Ulrich Pinkall 1996]

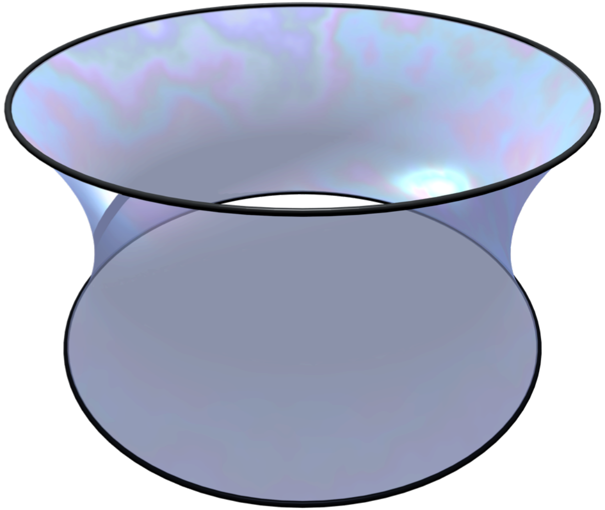
# A zoo of discrete nets



# Minimal surfaces

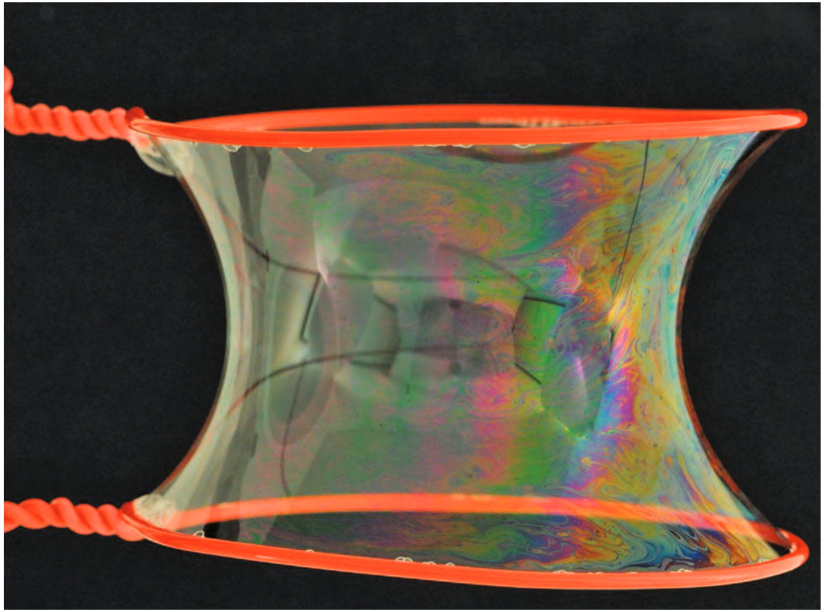
Local minima for the Plateu problem





[ Pictures taken from Emanuele Paolini's Minimal Surfaces page,
The surfaces were generated with "surf" and rendered with povray]

# Minimal surfaces

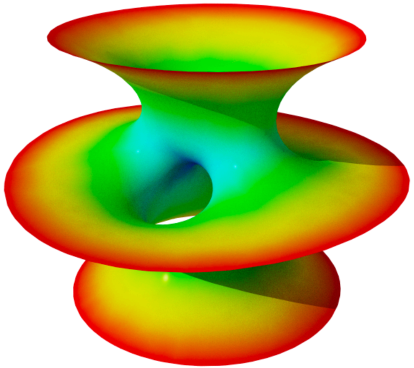
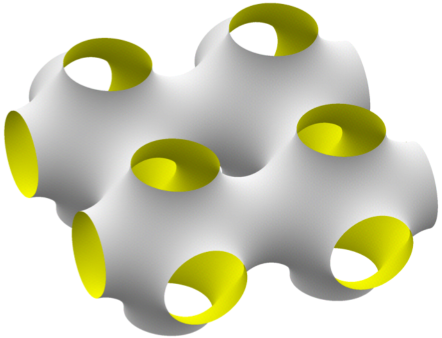
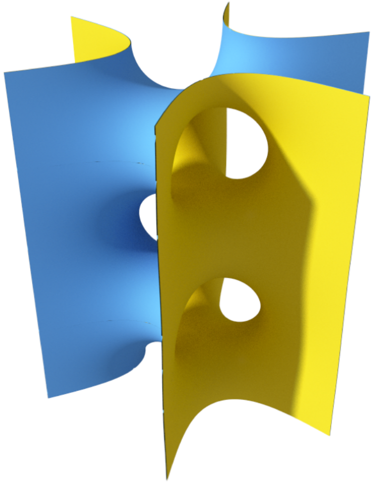


[ A new age of minimal surfaces – Joaquín Pérez]

# Minimal surfaces

- Gradient of area functional vanishes

$$H = 0$$

# Curvature and Area: Steiner's Formula

Offset surfaces

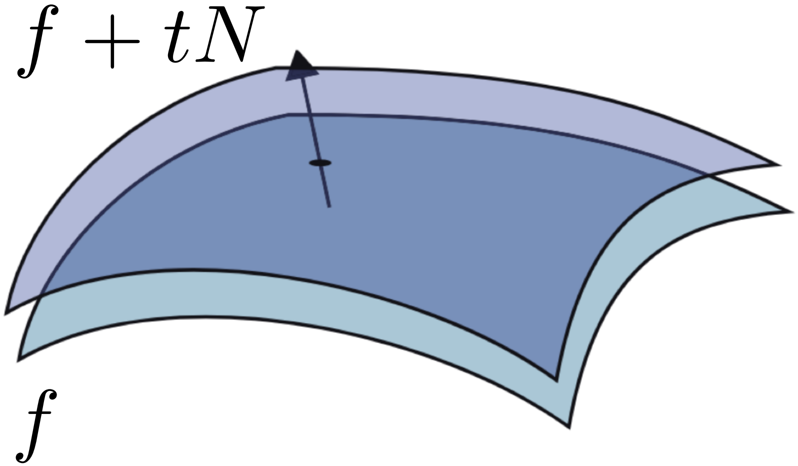



Image taken from "Discrete differential geometry of curves and surfaces" , Tim Hoffmann 2009.

# Curvature and Area: Steiner's Formula

Offset surfaces

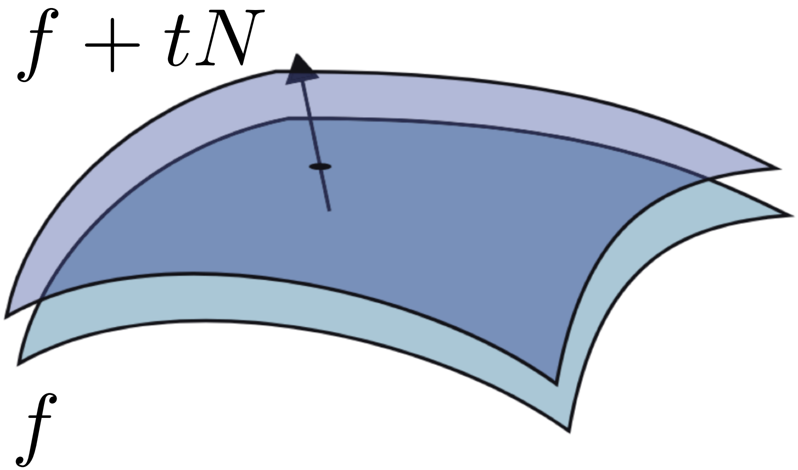



Image taken from "Discrete differential geometry of curves and surfaces" , Tim Hoffmann 2009.

Area of $A(f + tN)$

$$A(f) + 2t \int H(f) + t^2 \int K(f)$$

# Curvature and Area: Steiner's Formula

Offset surfaces

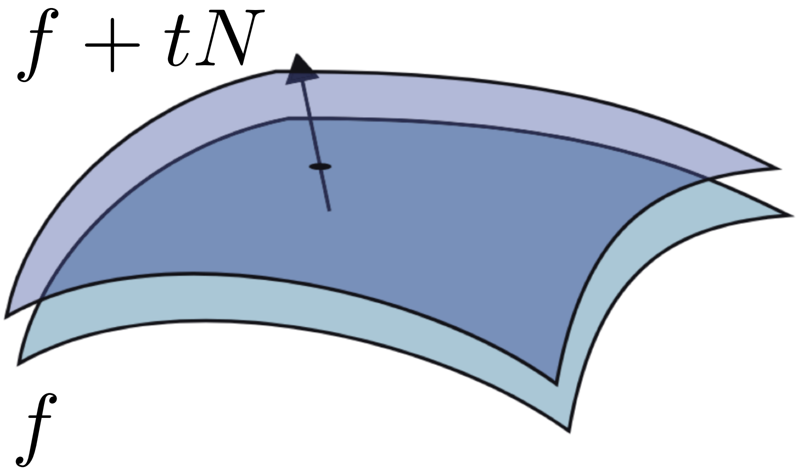



Image taken from "Discrete differential geometry of curves and surfaces" , Tim Hoffmann 2009.

Area of $A(f + tN)$

Quadratic form

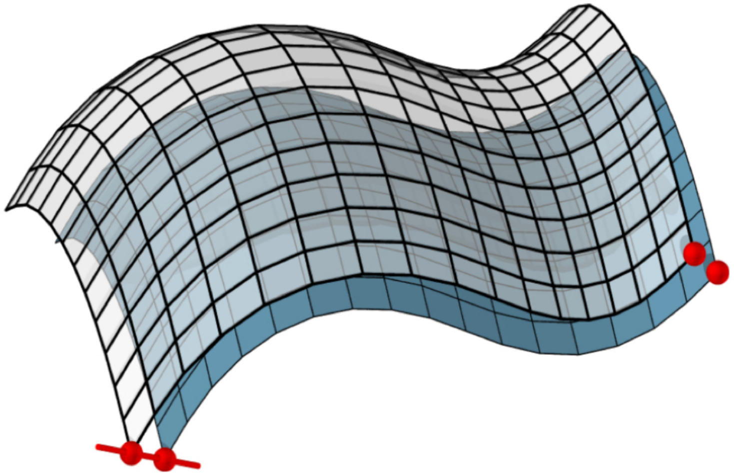
$$A(f) + 2t \int H(f) + t^2 \int K(f)$$

Linear coefficient

Quadratic coefficient

# Discretization

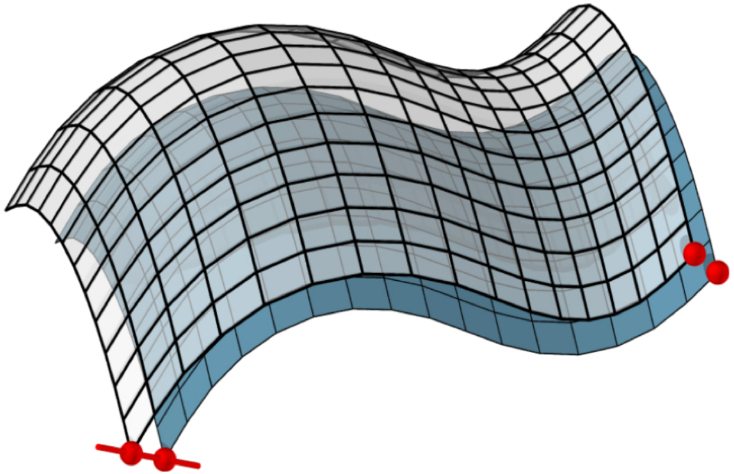
## Area Change by Normal Push in Constant Distance



[Pottmann et al. 2007]
[Bobenko et al. 2009]

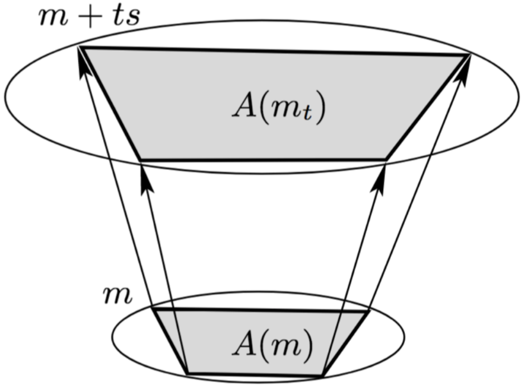
# Discretization

## Area Change by Normal Push in Constant Distance



Quadratic form

Linear and quadratic coefficients define $H, K$




[Pottmann et al. 2007]
[Bobenko et al. 2009]

[A curvature theory for discrete surfaces based on mesh parallelity, Bobenko et al. 2010]

# Discrete Minimal Surfaces

$$H = 0$$

[A curvature theory for discrete surfaces based on mesh parallelity, Bobenko et al. 2010]

[Discrete isothermic surfaces, Bobenko Alexander and Ulrich Pinkall 1996]

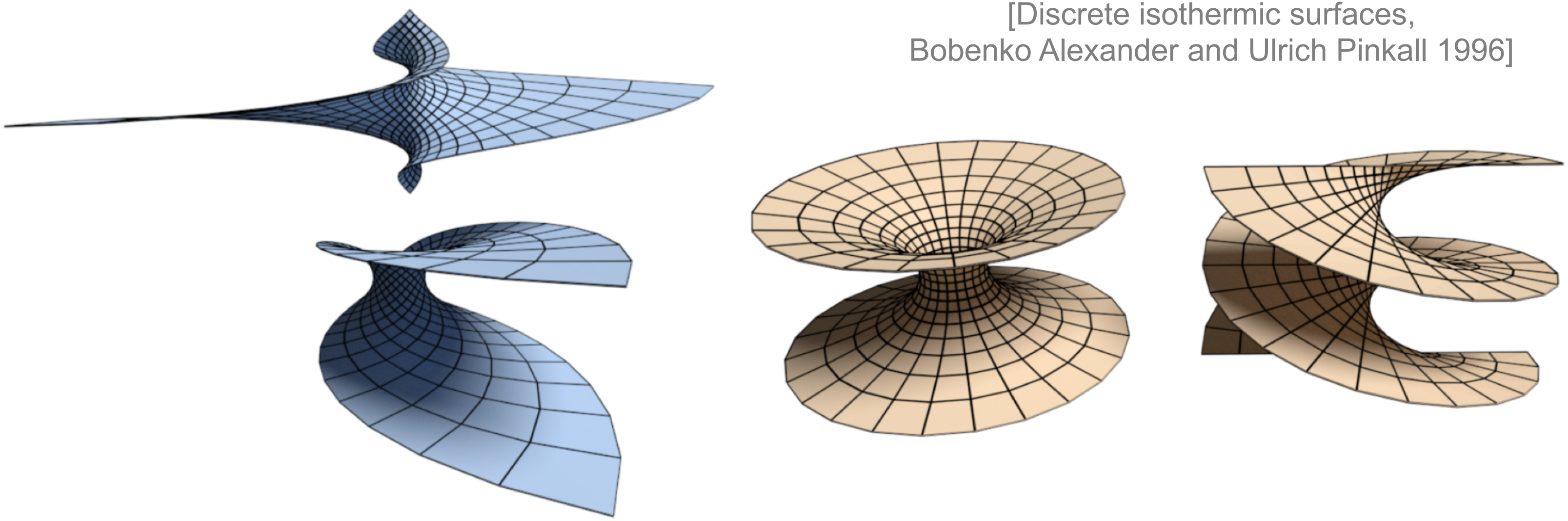
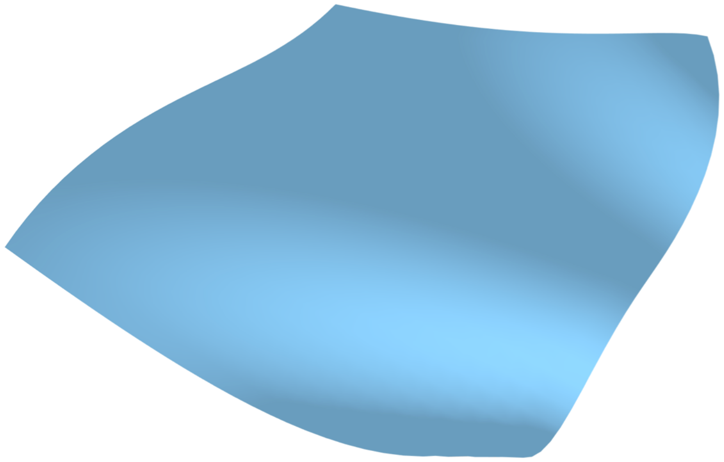


Image taken from [Hoffmann et al. 2014]

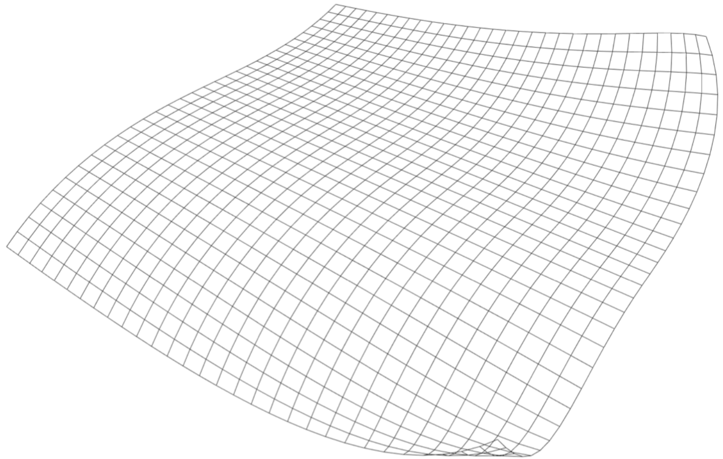
# Summary

# A discrete theory

Smooth: Differential Geometry



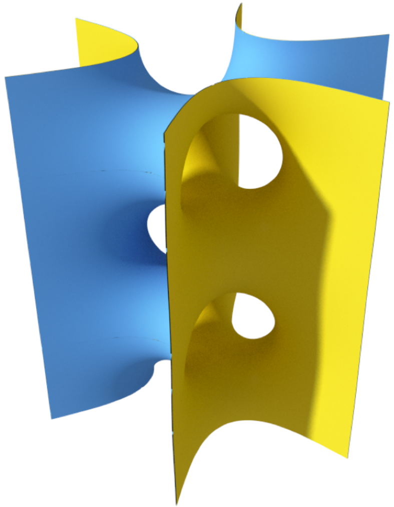
Discrete : Discrete Differential Geometry



*Discretize the whole theory, not just the equations*

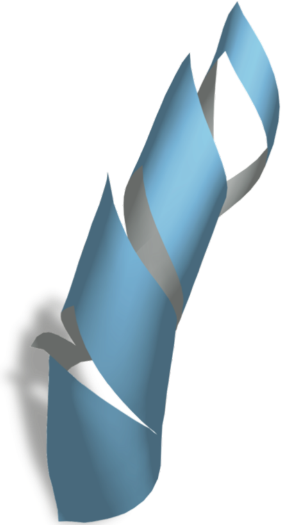
# Focus on specific parameterization/meshing

Orthogonal Asymptotic



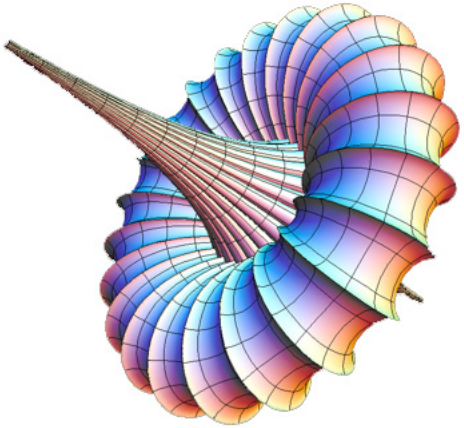
Minimal surfaces

Orthogonal Geodesics



Developable

Asymptotic Chebyshev



Pseudospherical

Virtual Math Museum, http://virtualmathmuseum.org/index.html

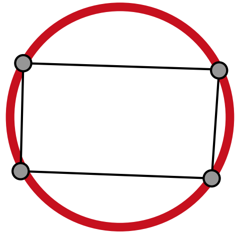
# Simple

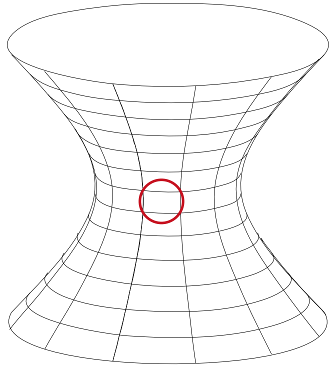
## Smooth

$$n_x \parallel f_x, n_y \parallel f_y$$

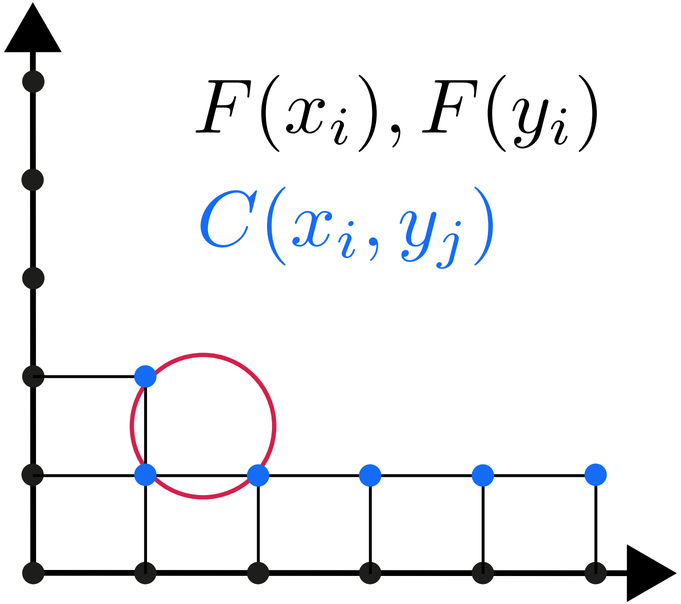
## Discrete

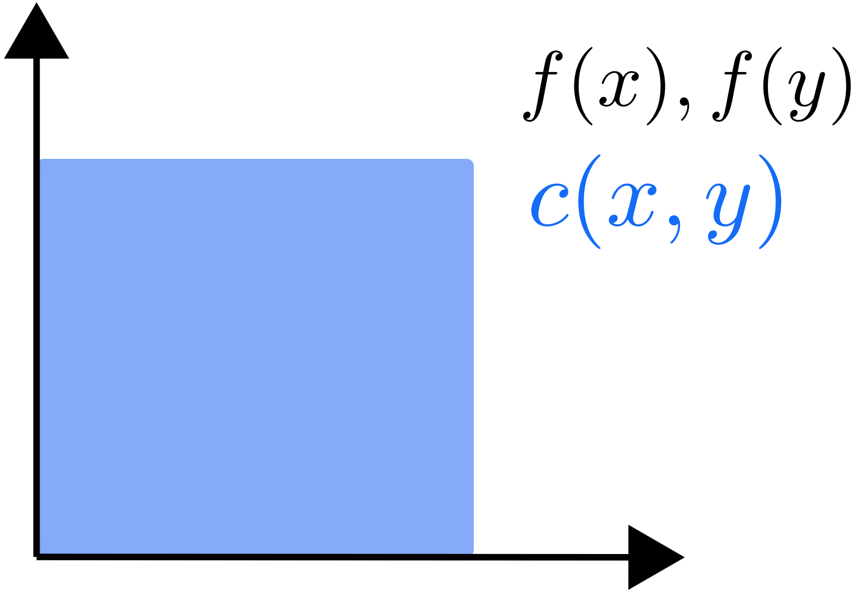
Quads are circular





# Preserves structure

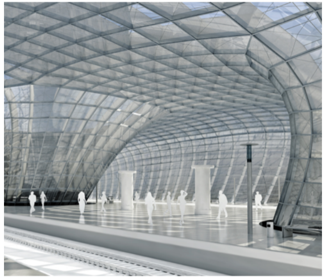


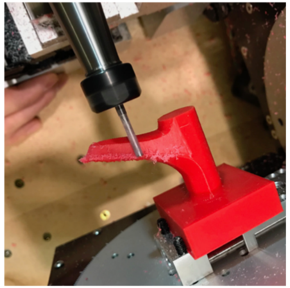
# Applications

*Nur Alem*, Astana Kazakhstan

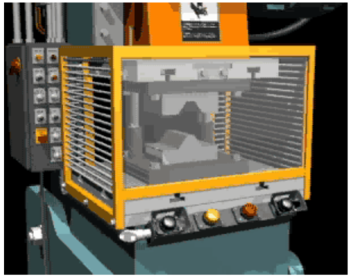
Disney Concert Hall,
LA, Frank Gehry



[Liu et al . 2007]



[Stein et al. 2018]



Stamping, wikipedia

[Akash Garg et al. 2014]

# Thank you!

## Questions?