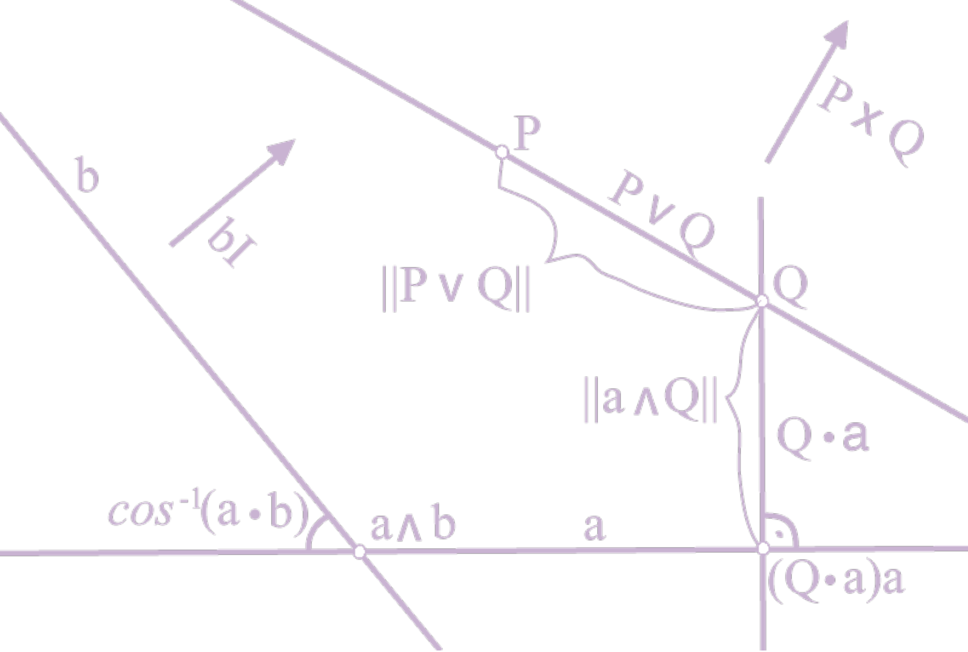


P
P × Q
b
bI
P ∨ Q
Q
||P ∨ Q||
||a ∧ Q||
Q • a
cos⁻¹(a • b)
a ∧ b
a
(Q • a)a

Projective geometric algebra:

A Swiss army knife for graphics and games

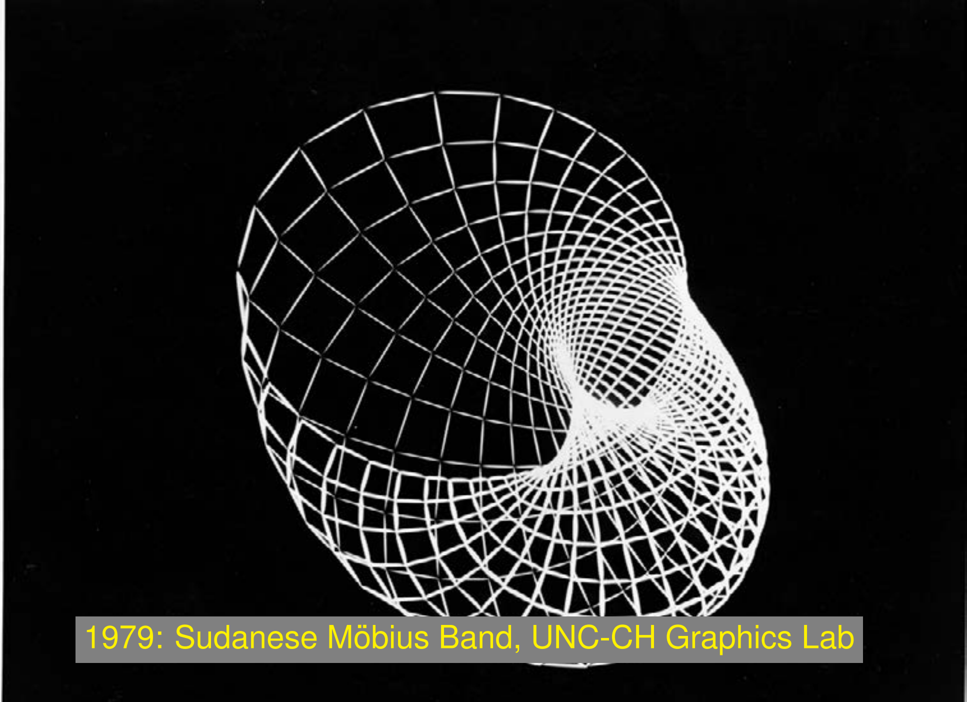
Charles Gunn

19.06.2016

Geometry Summit Summer School

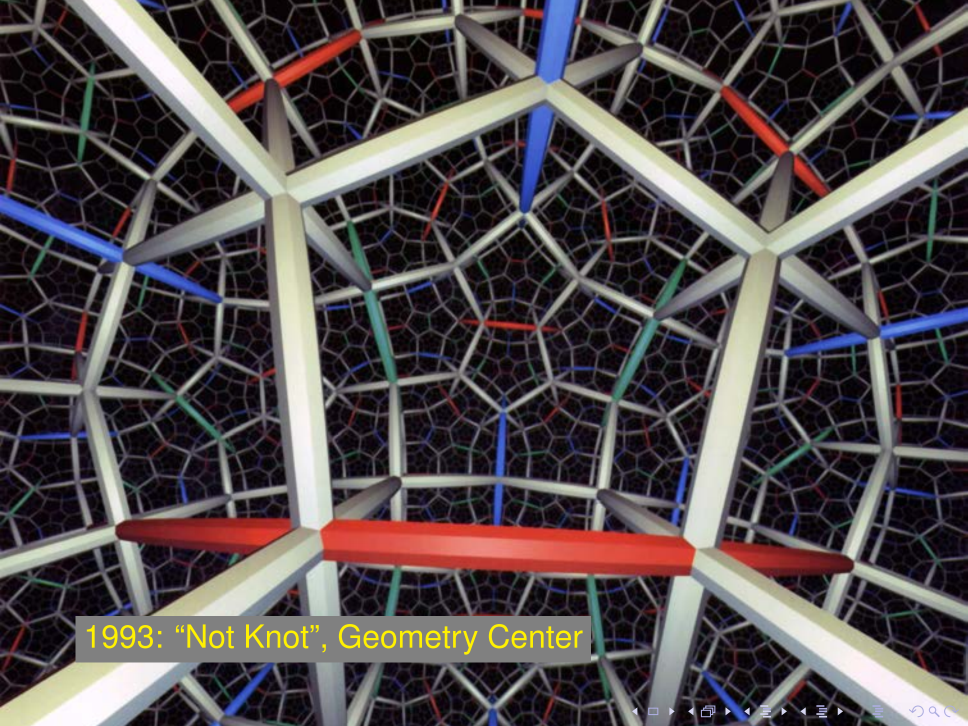
FU-Berlin


22·FJ·18
“Old Timer”



1979: Sudanese Möbius Band, UNC-CH Graphics Lab

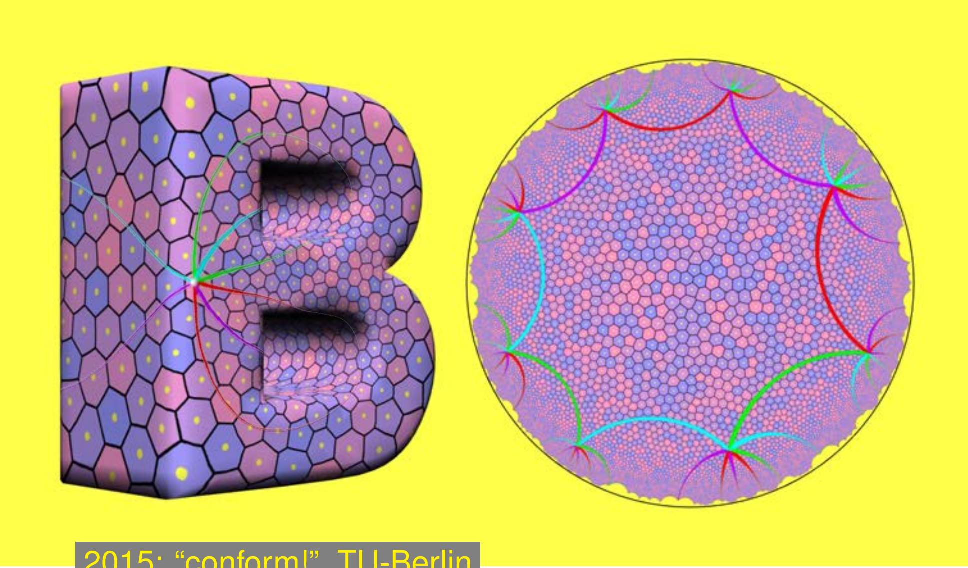
1987: “Red’s Dream”, Pixar



1993: “Not Knot”, Geometry Center

2004-2016, jReality developer, TU-Berlin

2006: “The Borromean Rings”, TU-Berlin



2015: “conform!”, TU-Berlin

Standard toolkit for euclidean geometry

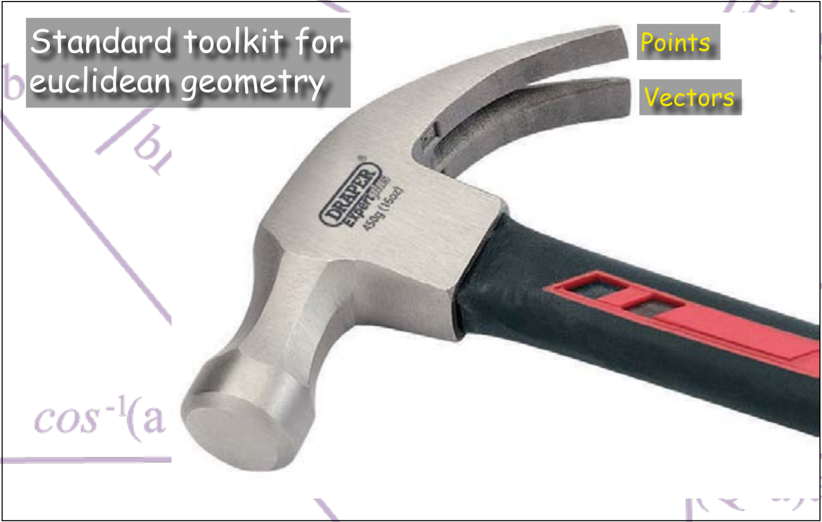
By euclidean geometry we mean

the geometry of euclidean space $\mathbf{E}^n$.

not euclidean vector space $\mathbb{R}^n$.

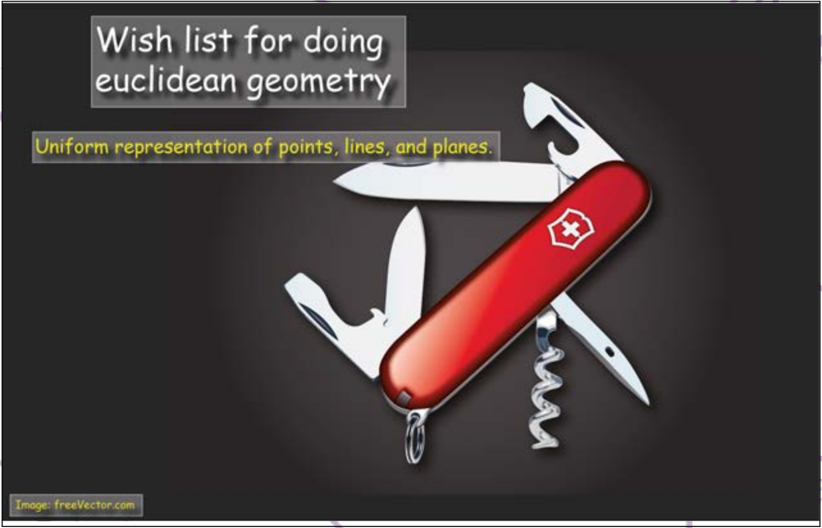
Standard toolkit for euclidean geometry

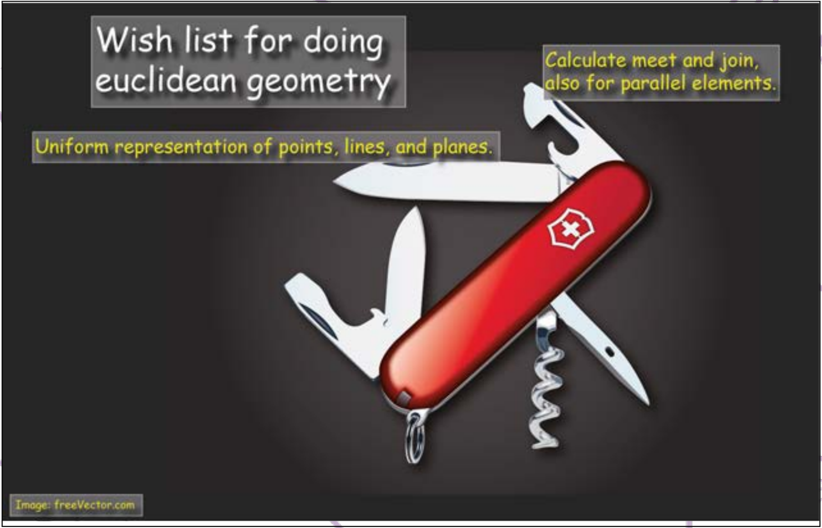
Points

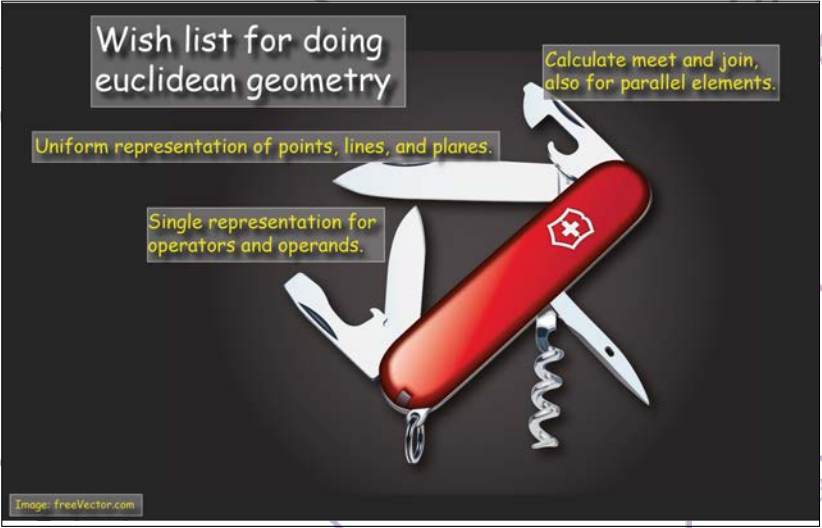

Standard toolkit for
euclidean geometry
Points
Vectors

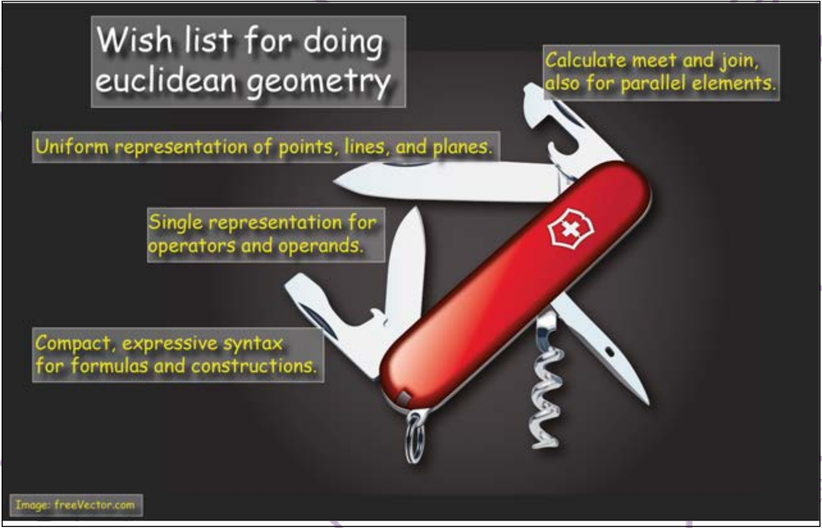

Standard toolkit for euclidean geometry
Points
Vectors
Matrices
DRAPER
450g (16oz)

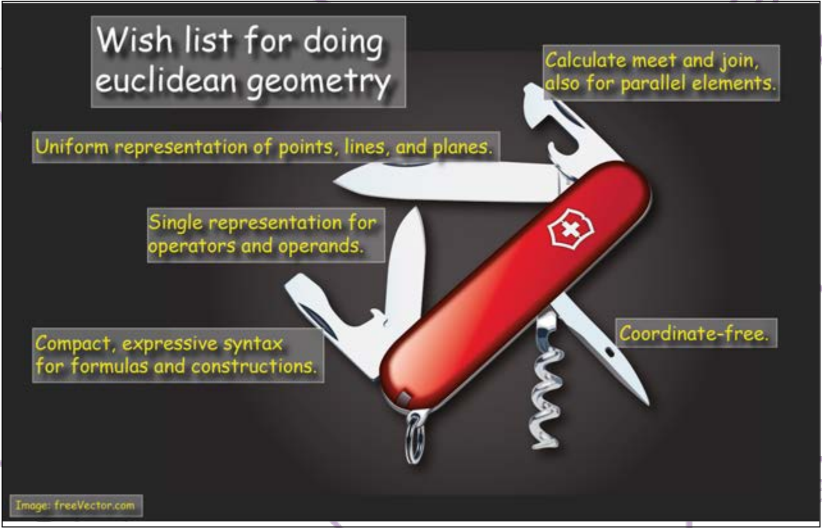

Wish list for doing
euclidean geometry
Image: freeVector.com


Wish list for doing
euclidean geometry
Uniform representation of points, lines, and planes.
Image: freeVector.com


Wish list for doing euclidean geometry
Calculate meet and join, also for parallel elements.
Uniform representation of points, lines, and planes.
Image: freeVector.com


Wish list for doing euclidean geometry
Calculate meet and join, also for parallel elements.
Uniform representation of points, lines, and planes.
Single representation for operators and operands.
Image: freeVector.com


Wish list for doing euclidean geometry
Calculate meet and join, also for parallel elements.
Uniform representation of points, lines, and planes.
Single representation for operators and operands.
Compact, expressive syntax for formulas and constructions.
Image: freeVector.com


Wish list for doing euclidean geometry
Calculate meet and join, also for parallel elements.
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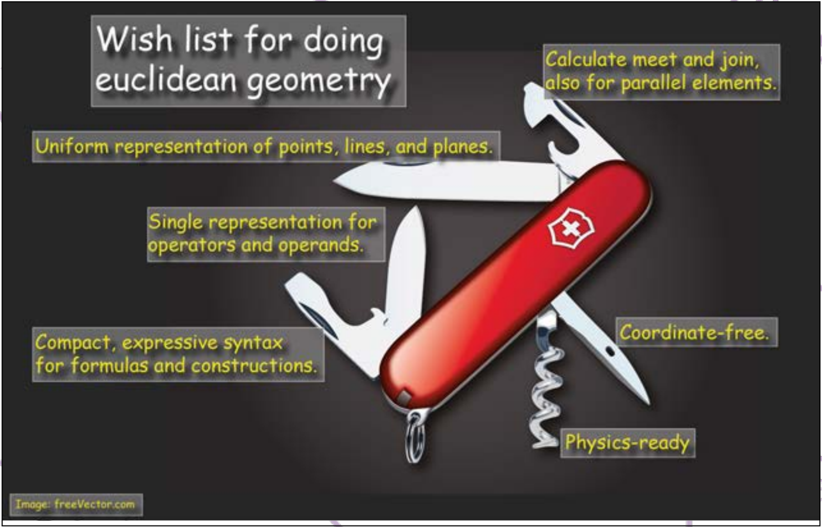
Compact, expressive syntax for formulas and constructions.
Coordinate-free.
Image: freeVector.com


Wish list for doing euclidean geometry
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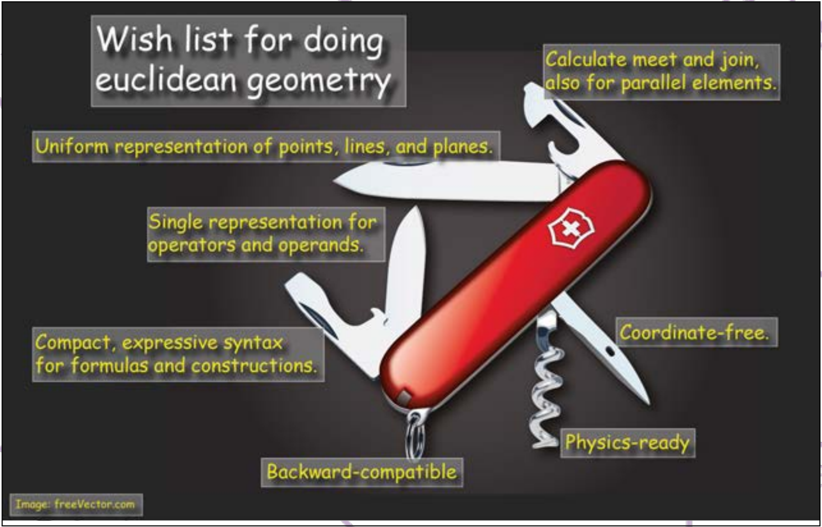
Coordinate-free.
Physics-ready
Image: freeVector.com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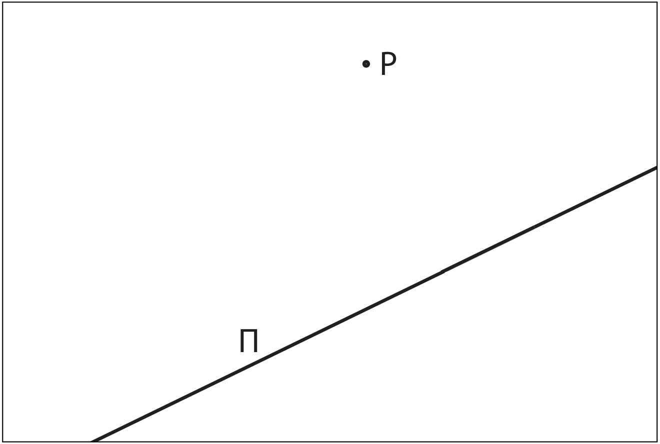
Physics-ready
Backward-compatible
Image: freeVector.com

Main Idea

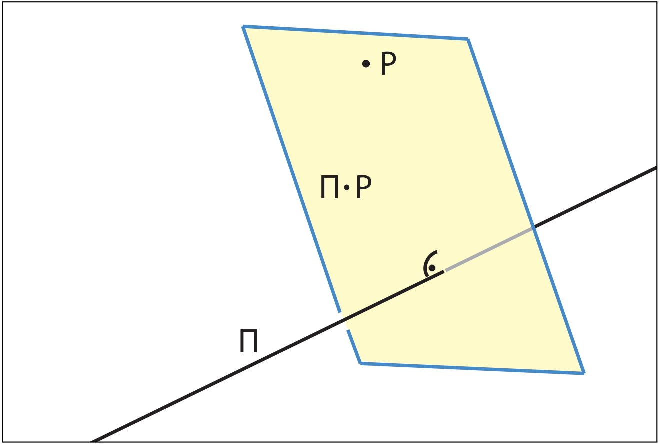
The main idea is to represent geometric primitives (points, lines, planes) as numbers which can be multiplied with each other using a geometric product.

Example 1: A geometric construction



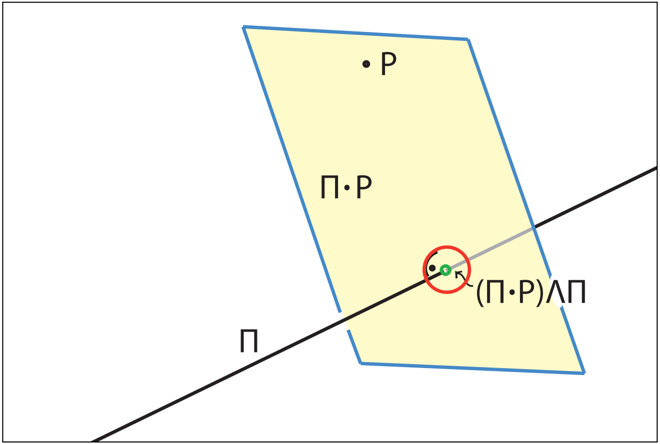
Task: Given a point **P** and a line Π in $\mathbf{E}^3$, find the unique line through **P** perpendicular to Π.

Example 1: A geometric construction



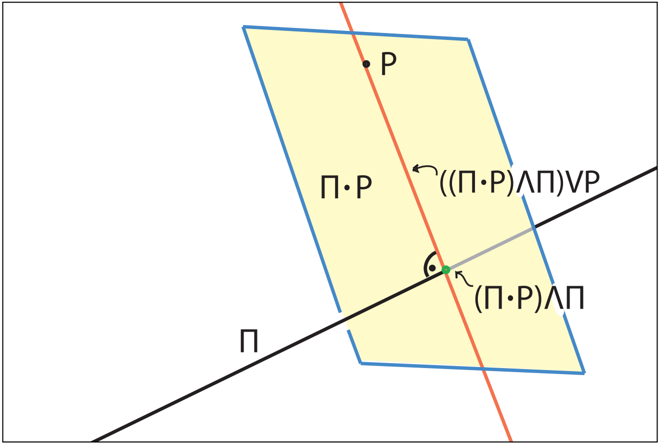
Step 1: $\Pi \cdot \mathbf{P}$ is the plane through **P** perpendicular to Π.

Example 1: A geometric construction



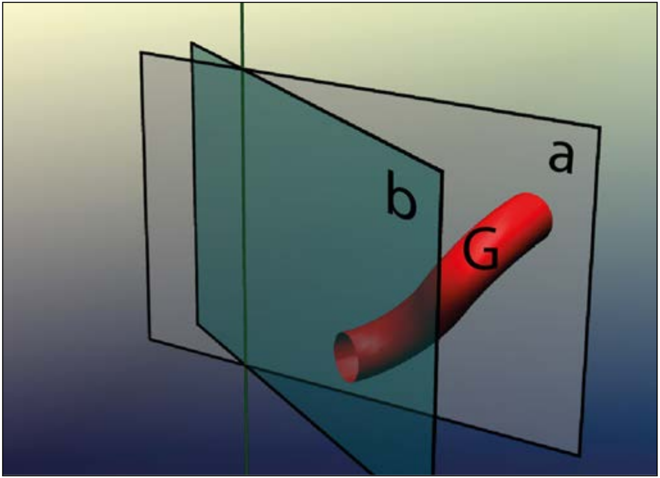
Step 2: $(\Pi \cdot \mathbf{P}) \wedge \Pi$ is the intersection of this plane with Π.

Example 1: A geometric construction



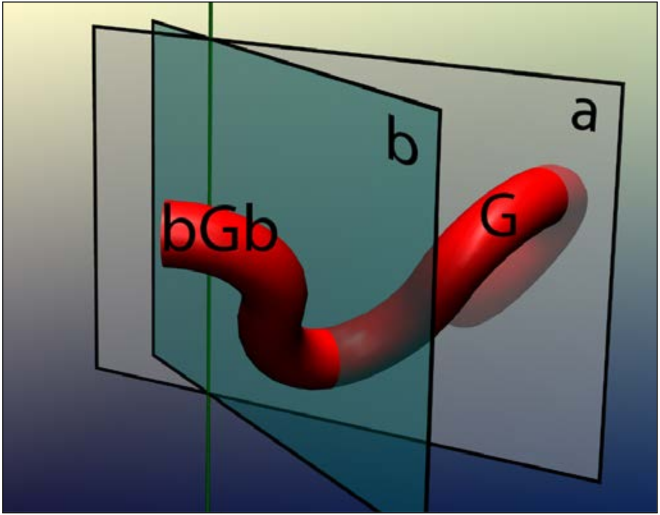
Step 3: $((\Pi \cdot \mathbf{P}) \wedge \Pi) \vee \mathbf{P}$ is the joining line of this point with **P**.

Example 2: A kaleidoscope



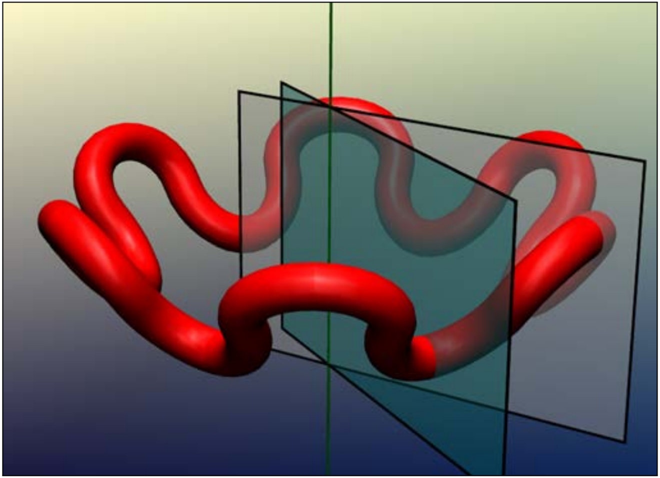
Task: Given mirror planes **a** and **b** and some geometry **G**, represent the kaleidoscope generated by the mirrors and **G**.

Example 2: A kaleidoscope



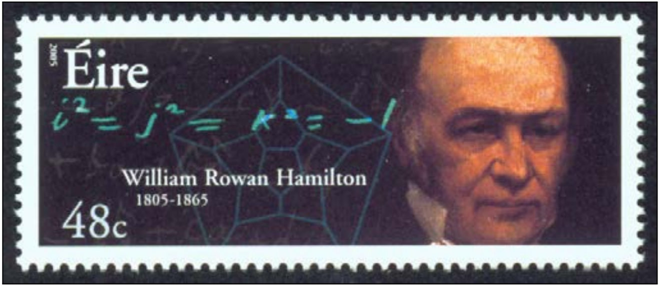
Step 1: **bGb** is the reflection of **G** in **b**, **aGa** the reflection in **a.**

Example 2: A kaleidoscope



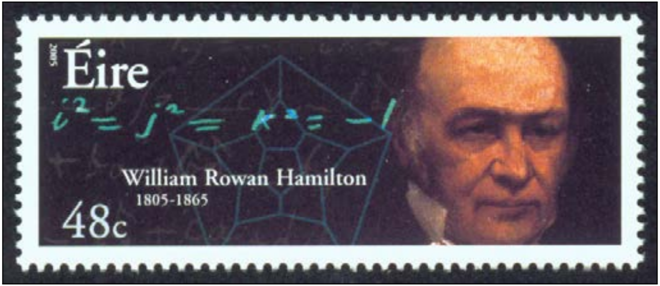
Solution: Form “sandwiches” **aGa**, **bGb**, **abGba**, **abaGaba** etc., subject to the relation $(\mathbf{ab})^6 = 1$.

Antecedents ...



QUATERNIONS: An algebra for $\mathbb{R}^3$

Antecedents ...



QUATERNIONS: An algebra for $\mathbb{R}^3$

Quaternions (1843)

- **Imaginary quaternions.** $\mathbb{IH} : (x, y, z) \in \mathbb{R}^3 \leftrightarrow x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
- **Unit quaternions.** $\mathbb{U} := \{\mathbf{g} \in \mathbb{H} \mid \mathbf{g}\bar{\mathbf{g}} = 1\}$.
- **"Geometric" product.** For $\mathbf{g},\mathbf{h} \in \mathbb{IH}$,

$$\mathbf{gh} = -\mathbf{g} \cdot \mathbf{h} + \mathbf{g} \times \mathbf{h}$$

- **Exponential map** $\mathbb{IH} \to \mathbb{U}$. $\mathbf{g} \in \mathbb{U}$ can be written as $\mathbf{g} = e^{t\mathbf{v}} (= \cos t + \sin t\mathbf{v})$ with $\mathbf{v} \in \mathbb{IH}$ and $\mathbf{v}^2 = -1$.
- **Rotations as sandwiches.** For $\mathbf{x} \in \mathbb{R}^3$ and $\mathbf{g} \in \mathbb{U}$, $\mathbf{gx}\bar{\mathbf{g}}$ is a rotation of $\mathbf{x}$ around the axis $\mathbf{v}$ through an angle $2t$.
- **ODE's for Euler top.**

$$\dot{\mathbf{g}} = \mathbf{g}\mathbf{V}_c$$

$$\dot{\mathbf{M}}_c = \frac{1}{2}(\mathbf{V}_c\mathbf{M}_c - \mathbf{M}_c\mathbf{V}_c)$$

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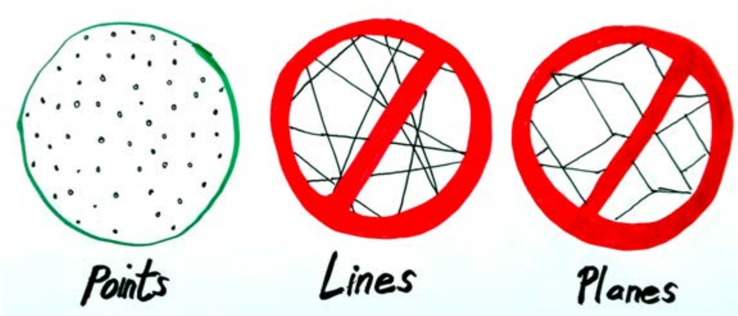
Quaternions (1843)

BUT:

Quaternions don't allow for representing lines or planes, only points.



Quaternions (1843)

AND:

Quaternions don't allow for representing

translations,

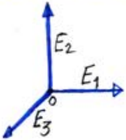
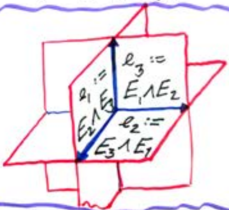
only rotations around the origin.

Grassmann algebra

Hermann Grassmann (1807-1877)

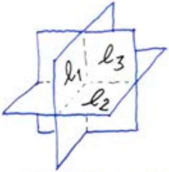
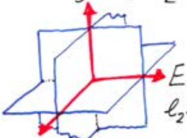
Ausdehnungslehre (1844).

GRASSMANN ALGEBRA $\Lambda\mathbb{R}^3$

Generators	$\{1\}$	$\{E_1, E_2, E_3\}$	$\{e_1, e_2, e_3\}$	$\{I\}$
?				$I := E_1 \wedge E_2 \wedge E_3$
Type	scalars	vectors	oriented planes thru origin	pseudo-scalar
Grade	0	1	2	3
Dim'n	1	3	3	1

Λ is the join operator of oriented subspaces.

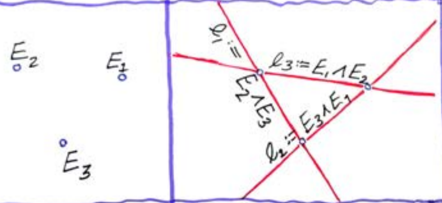
GRASSMANN ALGEBRA $\Lambda\mathbb{R}^{3*}$

Generators	$\{1\}$	$\{e_1, e_2, e_3\}$	$\{E_1, E_2, E_3\}$	$\{I\}$
?				$I := e_1 \wedge e_2 \wedge e_3$
Type	scalars	oriented planes thru O	vectors	pseudo-scalar
Grade	0	1	2	3
Dim'n	1	3	3	1

$\wedge$ is the **meet** operator of oriented subspaces.

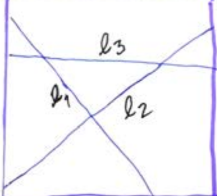
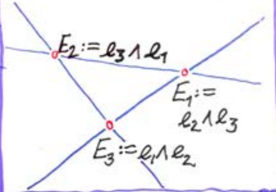
GRASSMANN ALGEBRA $P(\Lambda R^3)$

Generators	$\{1\}$	$\{E_1, E_2, E_3\}$	$\{e_1, e_2, e_3\}$	$\{I\}$
?				$I := E_1 \wedge E_2 \wedge E_3$
Type	scalars	points in $\mathbb{RP}^2$	lines in $\mathbb{RP}^2$	pseudo-scalar
Grade	0	1	2	3
Dim'n	1	3	3	1



Λ is the join operator of oriented subspaces.

GRASSMANN ALGEBRA $P(\Lambda \mathbb{R}^{3*})$

Generators	$\{1\}$	$\{e_1, e_2, e_3\}$	$\{E_1, E_2, E_3\}$	$\{I\}$
?		 e_3, e_1, e_2	 $E_2 := e_3 \wedge e_1$, $E_1 := e_2 \wedge e_3$, $E_3 := e_1 \wedge e_2$	$I := e_1 \wedge e_2 \wedge e_3$
Type	scalars	lines in $\mathbb{RP}^2$	points in $\mathbb{RP}^2$	pseudo-scalar
Grade	0	1	2	3
Dim'n	1	3	3	1

$\wedge$ is the **meet** operator of oriented subspaces.

Grassmann algebra

BUT:

Grassmann algebra doesn't include
inner or cross products,
only outer product
(no metric).

William Clifford

William Clifford (1845-1879)

Inventor of biquaternions and of geometric algebra

Biquaternions (1873)

Clifford's first great discovery, biquaternions, does for $\mathbf{E}^3$ what $\mathbb{H}$ does for $\mathbb{R}^3$.

- Biquaternions: $\mathbf{g} + \epsilon\mathbf{h}$ where $\epsilon^2 \in \{1, -1, 0\}$.
- When $\epsilon^2 = 0$, called dual quaternions $\mathbb{DH}$.
- All the listed features of $\mathbb{H}$ generalize to $\mathbb{DH}$.
  - "Geometric" product.
  - Exponential map from imaginary $\mathbb{DH}$ to unit $\mathbb{DH}$.
  - Unit $\mathbb{DH}$: rotations and translations as sandwiches.
  - ODE's for free top.
- But, like the quaternions, it does not include meet and join operators.

Geometric algebra (1878)

The main idea: add an inner product to a Grassmann algebra.

- An inner product $\mathbf{a} \cdot \mathbf{b}$ is a symmetric bilinear form defined on 1-vectors.
- It is characterized by its signature, a triple (p, n, z), telling how many basis vectors square to 1, -1, and 0 (resp.).
- Define a geometric product on 1-vectors by:

$$\mathbf{ab} := \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$$

- It can be extended to an associative product on the whole Grassmann algebra to produce a geometric algebra.
- It is written $\mathbb{R}_{p,n,z}$ or $\mathbb{R}^*_{p,n,z}$ or $\mathbf{P}(\mathbb{R}_{p,n,z})$ or $\mathbf{P}(\mathbb{R}^*_{p,n,z})$, depending on the base Grassmann algebra.

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Geometric algebra (1878)

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Geometric algebra for euclidean geometry

- Exercise: $\mathbb{R}_{3,0,0}$ (or $\mathbb{R}^*_{3,0,0}$) is the desired geometric algebra for euclidean vector space $\mathbb{R}^3$.
- Exercise: $\mathbb{R}^+_{3,0,0} \simeq \mathbb{H}$.
- Non-euclidean geometies. $\mathbf{P}(\mathbb{R}_{3,0,0})$ is a geometric algebra for the 2-sphere, and $\mathbf{P}(\mathbb{R}_{2,1,0})$ for the hyperbolic plane.
- But a GA for $\mathbf{E}^n$ remained elusive 100 years after Clifford's early death (1879).

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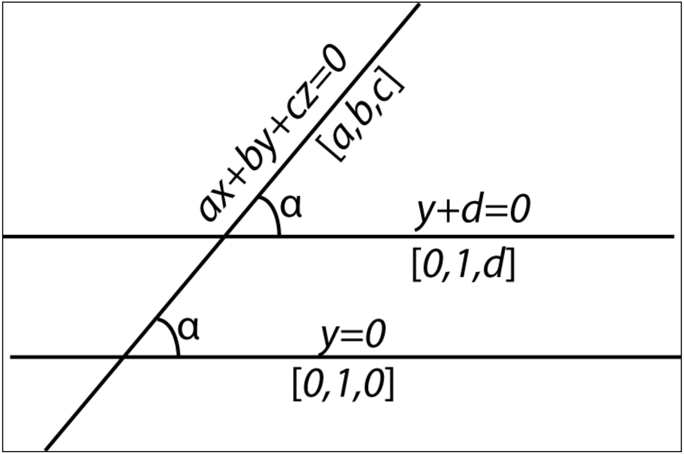
Painting: Henry Rheam


After a 100-year sleep...
Painting: Henry Rheam


Jon Selig (2000)
After a 100-year sleep...
Painting: Henry Rheam

Example: $n = 2$, the euclidean plane



$\langle [a_1, b_1, c1], [a_2, b_2, c_2] \rangle = a_1 a_2 + b_1 b_2 = \cos\alpha$

The correct GA is thus $\mathbf{P}(\mathbb{R}^*_{2,0,1})$.

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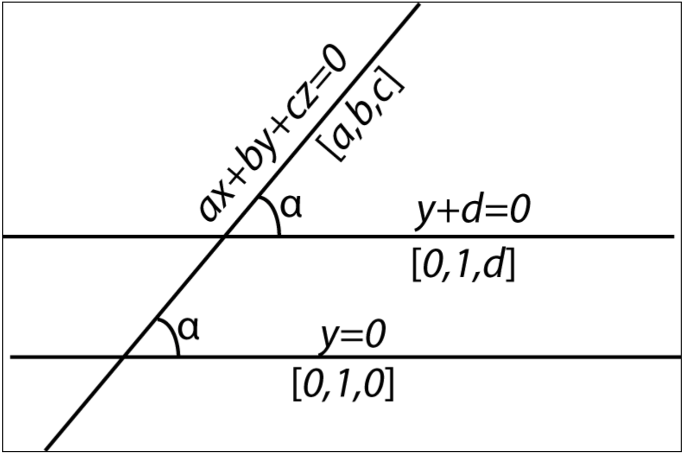

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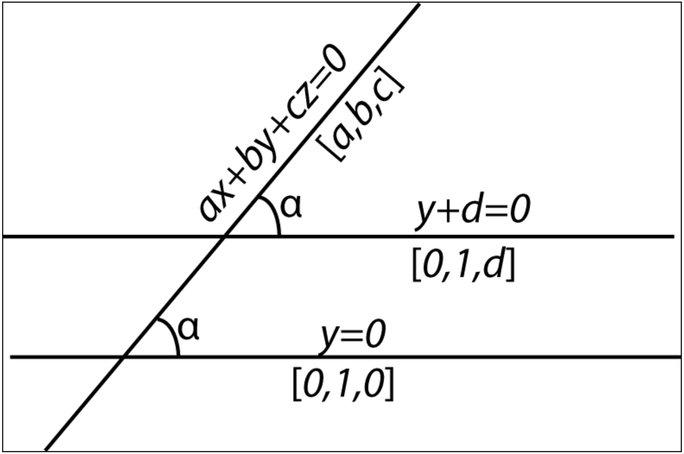

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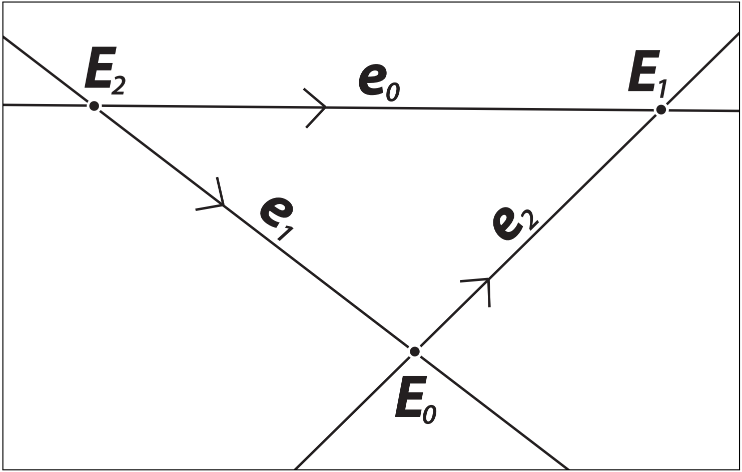
Basis vectors for $\mathbf{P}(\mathbb{R}^*_{2,0,1})$



Note: We have renamed $\mathbf{e}_3$ as $\mathbf{e}_0$ and $\mathbf{E}_3$ as $\mathbf{E}_0$.

Multiplication table for $\mathbf{P}(\mathbb{R}^*_{2,0,1})$

$$\mathbf{E}_0 := \mathbf{e}_1\mathbf{e}_2, \quad \mathbf{E}_1 := \mathbf{e}_2\mathbf{e}_0, \quad \mathbf{E}_2 := \mathbf{e}_0\mathbf{e}_1, \quad \mathbf{I} := \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2$$

	$\mathbf{1}$	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{E}_0$	$\mathbf{E}_1$	$\mathbf{E}_2$	$\mathbf{I}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{E}_0$	$\mathbf{E}_1$	$\mathbf{E}_2$	$\mathbf{I}$
$\mathbf{e}_0$	$\mathbf{e}_0$	0	$\mathbf{E}_2$	$-\mathbf{E}_1$	$\mathbf{I}$	0	0	0
$\mathbf{e}_1$	$\mathbf{e}_1$	$-\mathbf{E}_2$	$\mathbf{1}$	$\mathbf{E}_0$	$\mathbf{e}_2$	$\mathbf{I}$	$-\mathbf{e}_0$	$\mathbf{E}_1$
$\mathbf{e}_2$	$\mathbf{e}_2$	$\mathbf{E}_1$	$-\mathbf{E}_0$	$\mathbf{1}$	$-\mathbf{e}_1$	$\mathbf{e}_0$	$\mathbf{I}$	$\mathbf{E}_2$
$\mathbf{E}_0$	$\mathbf{E}_0$	$\mathbf{I}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$-\mathbf{1}$	$-\mathbf{E}_2$	$\mathbf{E}_1$	$-\mathbf{e}_0$
$\mathbf{E}_1$	$\mathbf{E}_1$	0	$\mathbf{I}$	$-\mathbf{e}_0$	$\mathbf{E}_2$	0	0	0
$\mathbf{E}_2$	$\mathbf{E}_2$	0	$\mathbf{e}_0$	$\mathbf{I}$	$-\mathbf{E}_1$	0	0	0
$\mathbf{I}$	$\mathbf{I}$	0	$\mathbf{E}_1$	$\mathbf{E}_2$	$-\mathbf{e}_0$	0	0	0

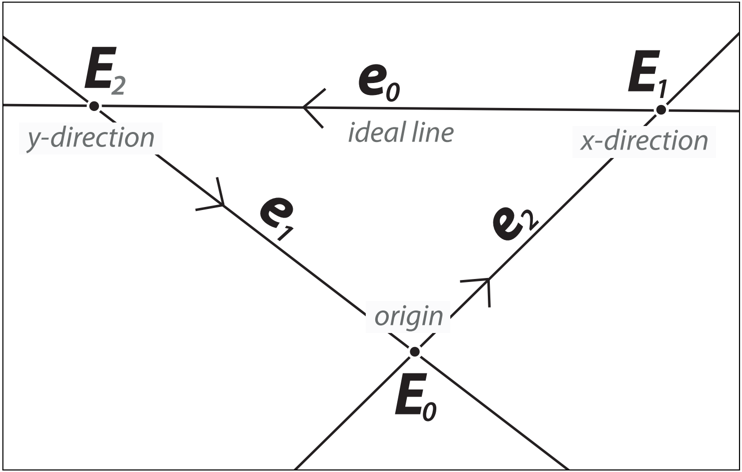
Geometric algebra notation

- $\langle \mathbf{X} \rangle_k$ is grade-projection operator: the grade-k part of $\mathbf{X}$.
- A *k-vector* satisfies $\mathbf{X} = \langle \mathbf{X} \rangle_k$.
- Write 1-vectors (lines) using small Roman letters $\mathbf{a}, \mathbf{b}$, etc.
- Write 2-vectors (points) using large Roman letters $\mathbf{A}, \mathbf{B}$, etc..

Euclidean and ideal elements

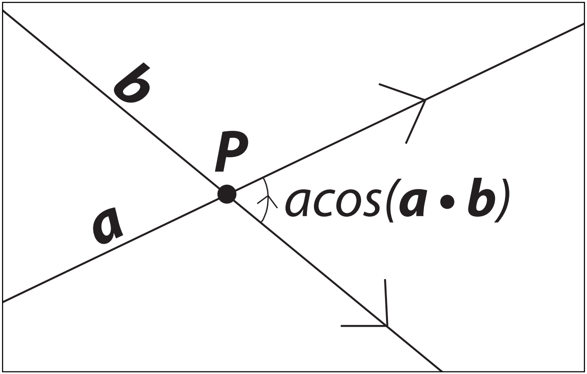
- For a k-vector $\mathbf{X} \in \mathbf{P}(\mathbb{R}^*_{2,0,1})$, $\mathbf{X}^2$ is a scalar.
  - A point or line satisfying $\mathbf{X}^2 \neq 0$ is euclidean.
  - A point or line satisfying $\mathbf{X}^2 = 0$ is ideal.
  - $\mathbf{e}_0$ is the ideal line, $\mathbf{E}_1$ and $\mathbf{E}_2$ are the ideal points in the x- and y-directions.
- Ideal points can be identified with free vectors.
- A euclidean line $\mathbf{a}$ can be normalized so $\|\mathbf{a}\| = 1$.
- A euclidean point $\mathbf{P}$ can be normalized so $\|\mathbf{P}\| = 1$.
- An ideal point $\mathbf{V}$ can be normalized so $\|\mathbf{V}\|_\infty = 1$.
  - Ideal norm $\| \; \|_\infty$ based on signature $(2, 0, 0)$ on $\mathbf{e}_0$.
  - $\Rightarrow \|(x, y, 0)\|_\infty = \sqrt{x^2 + y^2}$.
- We normalize everywhere we can!

Basis vectors for $\mathbf{P}(\mathbb{R}^*_{2,0,1})$



ab

Assume **a** and **b** are two normalized euclidean lines (1-vectors).



If **a** and **b** intersect in a normalized euclidean point **P**:

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} = \cos\alpha + (\sin\alpha)\mathbf{P}$$

where α is the oriented angle between the lines.

ab

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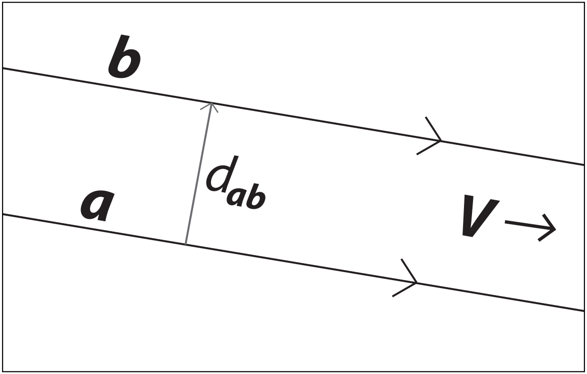

If **a** and **b** intersect in a normalized ideal point **V**:

$$\mathbf{ab} = 1 + d_{\mathbf{ab}}\mathbf{V}$$

where $d_{\mathbf{ab}}$ is the oriented distance between the lines.

ab

The formula correctly differentiates between the two cases and provides the appropriate weighting factor:

- an angle when the lines intersect, and
- a distance when they are parallel.

This interweaving of the euclidean and the ideal is a recurring theme in $\mathbf{P}(\mathbb{R}^*_{n,0,1})$.

ab

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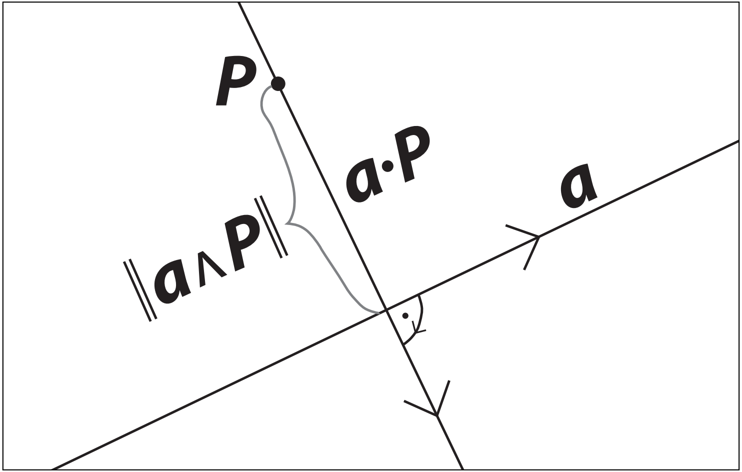
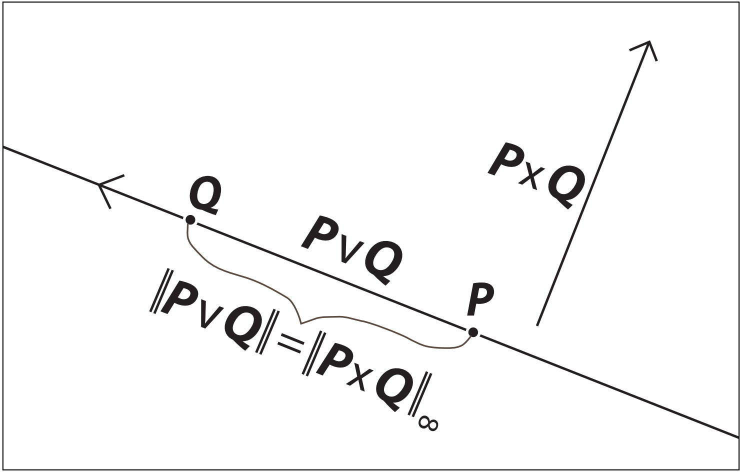
aP



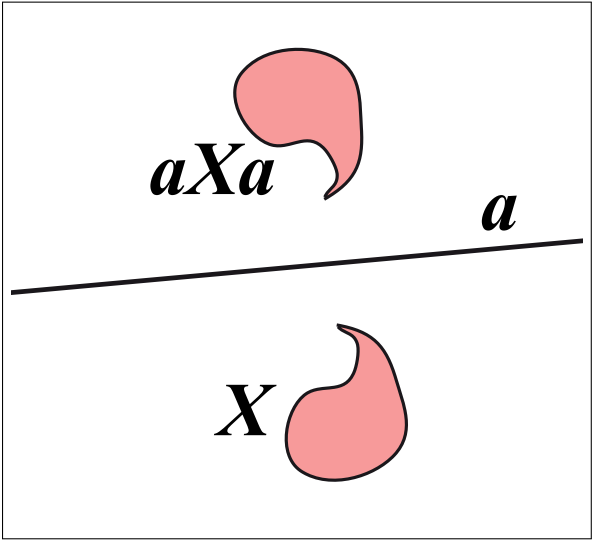
$$\mathbf{aP} = \langle \mathbf{aP} \rangle_1 + \langle \mathbf{aP} \rangle_3$$
$$= \mathbf{a} \cdot \mathbf{P} + d_{\mathbf{aP}} \mathbf{I}$$

PQ and P ∨ Q



$$\mathbf{PQ} = \langle \mathbf{PQ} \rangle_0 + \langle \mathbf{PQ} \rangle_2$$
$$= -1 + \mathbf{P} \times \mathbf{Q}$$

Reflections, rotations, translations, ...



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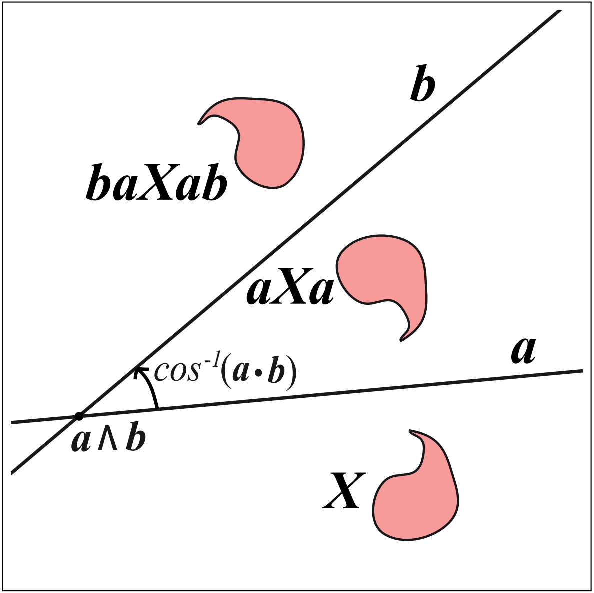

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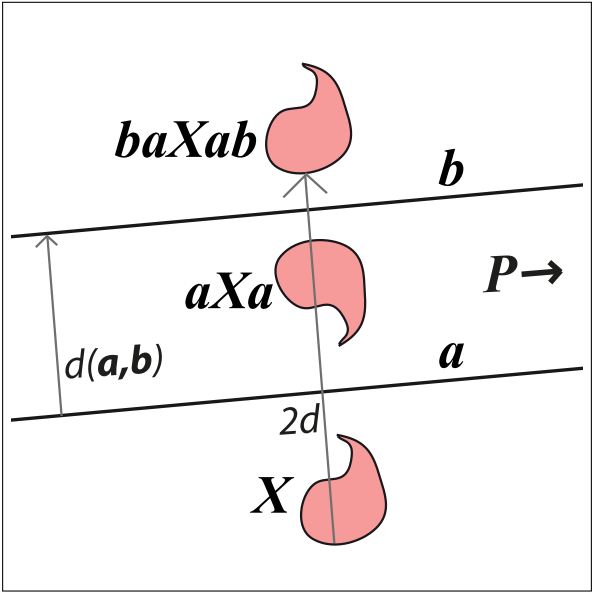


Isometries

More useful facts:

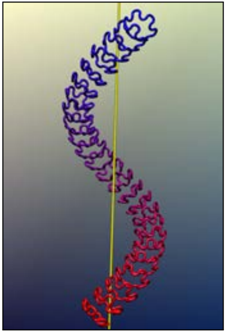
- $e^{t\mathbf{P}}$ produces a 1-parameter family of rotors.
  - They are rotations around the euclidean point **P**.
  - They are translations with direction perpendicular to the direction **P** for ideal **P**.
- $\mathbf{P}(\mathbb{R}^{*+}_{2,0,1})$ is isomorphic to the “planar quaternions”.

3D

For $\mathbf{E}^3$ the corresponding PGA is $\mathbf{P}(\mathbb{R}^*_{3,0,1})$.

- The even subalgebra $\mathbf{P}(\mathbb{R}^{*+}_{3,0,1})$ is isomorphic to $\mathbb{DH}$.
  - $\epsilon \in \mathbb{DH}$ maps to the pseudoscalar $\mathbf{I} \in \mathbf{P}(\mathbb{R}^*_{3,0,1})$.
- Thus, Clifford's two big discoveries are combined within $\mathbf{P}(\mathbb{R}^*_{3,0,1})$.
- Things are much more interesting since there are 2-vectors whose squares are not scalars: linear line complexes.

Example 3: Screw motions



Task: Given a line Σ in $\mathbf{E}^3$, represent the screw motion around Σ with given pitch.

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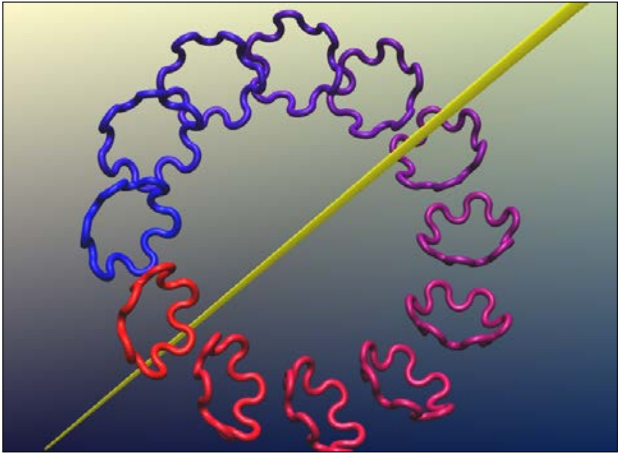

Step 1: The rotor given by $e^{t\Sigma}$ is a rotation: the sandwich $e^{t\Sigma}\mathbf{G}e^{-t\Sigma}$ rotates $\mathbf{G}$ around Σ thru angle $2t$.

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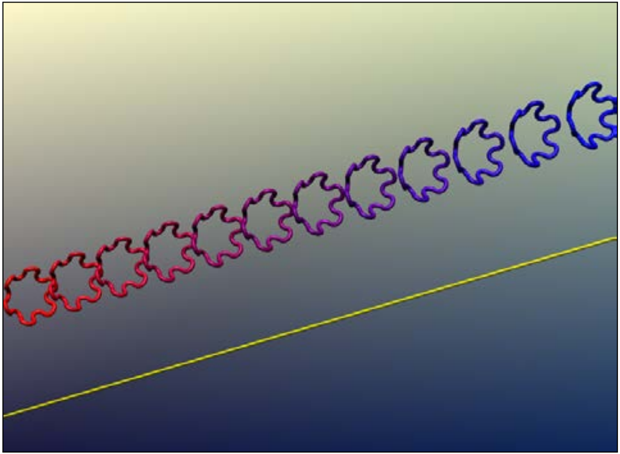

Step 2: The rotor given by $e^{d\Sigma \mathbf{I}}$ is a translation along Σ of distance $2d$ (a "rotation" around the polar line of Σ).

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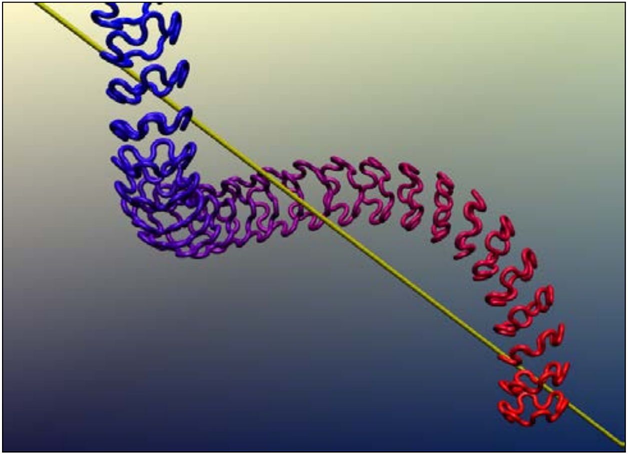
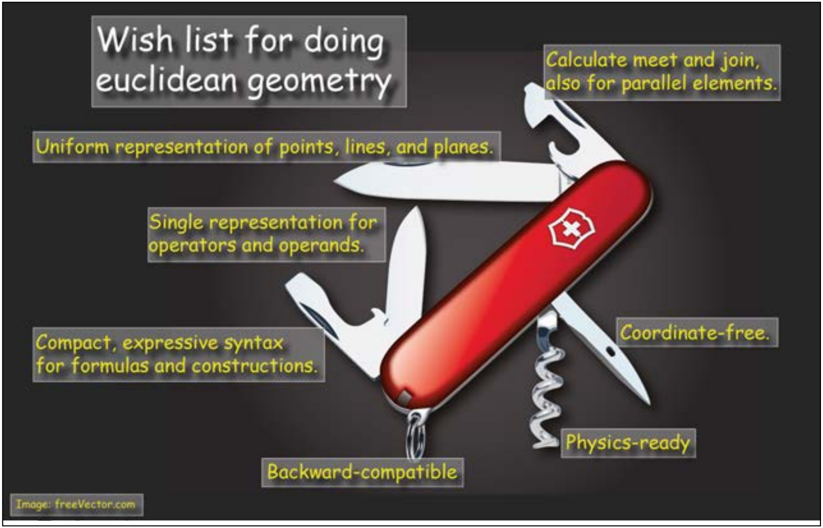


Step 3:The rotor given by $e^{(t+d\mathbf{I})\Sigma}$ is a screw motion combining these two motions, with pitch $d : t$.

Conclusion



Wish list for doing euclidean geometry
Calculate meet and join, also for parallel elements.
Uniform representation of points, lines, and planes.
Single representation for operators and operands.
Compact, expressive syntax for formulas and constructions.
Coordinate-free.
Physics-ready
Backward-compatible
Image: freeVector.com

Conclusion

Additional insights:

- Euclidean and ideal norms form an organic whole.
- Contains $\mathbb{H}$ and $\mathbb{DH}$ as subalgebras.
- Much remains to be discovered and worked out.
- Bonus: It's fully metric-neutral if you want to do spherical or hyperbolic geometry!

More information

- Author's copy: http://arxiv.org/abs/1411.6502, "Geometric algebras for euclidean geometry"
- Preprint: http://arxiv.org/abs/1501.06511, "Doing euclidean plane geometry using projective geometric algebra"
- These slides and related resources: http://page.math.tu-berlin.de/~gunn/gsumm2016
- Thank you for your attention!